A
TREATISE
ON

PROBLEMS OF
MAXIMA AND MINIMA

SOLVED BY ALGEBRA

BY
RAMCHUNDRA

Thoroughly revised, edited and rewritten
BY
DR. M.S. RANGACHARI

A PUBLICATION OF

THE ASSOCIATION OF
MATHEMATICS TEACHERS OF INDIA

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RAMCHUNDRA

(Published in 1859 A.D.)

Thoroughly revised, edited and rewritten

by

Dr. M.S. Rangachari

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The Association of
Mathematics Teachers of India

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BETWEEN US

Dear Reader,

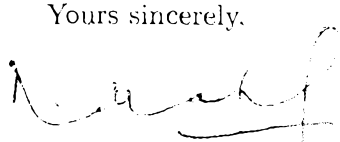
The Association of Mathematics Teachers of India has been publishing books to motivate the interested learner to identify novel approaches to known ideas and to keep the spirit of enquiry alive. In this book – A treatise on Problems of Maxima and Minima solved by Algebra by Ramchundra, – a pre-Ramanujan mathematician of India, true to the statement above novelty may be easily discernible. Prof. M.S.Rangachari, our senior member and retired Director, the Ramanujan Institute for Advanced Study in Mathematics, University of Madras, has revised the original book published in 1859 extensively and has presented the material in a form comprehensible to present day teachers and students.

We are very happy to place this book in the hands of admirers of the past glory of Indian mathematics and aspirants in talent competitors alike for going through the material for enjoyment in posing and solving problems different from the routine ones. We hope and trust that the Mathematics Olympiad students will find this book approach highly motivating and exciting in their pursuit of excellence.

I place on record our indebtedness to Prof.M.S.Rangachari for taking up this stupendous task. I also thank M/s Pagesmith for typesetting, M/s M.K.Graphics for plate making and M/s Souri Printers for printing and binding of the book in record time.

Looking forward to your response to this publication and with best wishes for enjoying mathematics,

Yours sincerely,



(M.MAHADEVAN)

EDITOR'S INTRODUCTION

This booklet is a rewritten and edited version of the booklet entitled "A treatise on problems of maxima and minima solved by algebra" by Ramchundra published in 1859 at London by WM. H. Allen & Co, 7, Leaden Hall Street under the superintendence of August De Morgan, then Professor of Mathematics in the University College, London. According to De Morgan, mathematicians of those days were not aware that calculus can be avoided in dealing with quite a few problems on maxima and minima. Ramchundra's work was quite original. It is to be noted that Ramchundra pertains to the pre-Ramanujan era in Indian mathematics. The present work testifies to the fact that there were persons capable of original thinking in mathematics passed on to them by the westerners even before Srinivasa Ramanujan.The present version is mainly intended to help students and teachers in schools and colleges in India and elsewhere and was prepared with an intention to update the terminology and remove obscurities in the original work without damaging the author's originality.

The editor thanks Drs. S.A.Settu and V.K.Krishnan for their help in correcting printer's proof and in removing a few obscurities. He also thanks Sri M.Mahadevan, the General Secretary of the Association for his cooperation during the printing of the booklet.

The life and work of Ramchundra were first exposed in the 'Indian Scientists', July 1929 issue published by G.A. Natesan & Co. in Chennai. The Editor wants to point out that the attention of the people interested in mathematics to Ramchundra's work was drawn in recent years by the late Professor C.Muses (lastly of Mathematics and Morphology, Editorial and Research Centre, 45911 Silver Ave., Sardia, BC, V2R1Z2, Canada) in an article in the Mathematics Intelligencer. It was Sri C.A.Reddi, a close friend of the editor and a keenly devoted student of Ramanujanelia, who brought Muses' article to the latter's notice and got him a copy of the work too. The Editor expresses his deep appreciation of Sri Reddi's efforts and hereby dedicates the present edition to the memory of the late Professor C.Muses.

M.S.RANGACHARI

ABOUT THE AUTHOR

Ramchundra, the author of the original booklet of which the present one is a thoroughly revised edition, was born at Panipet in 1821. His father was Sunderlal, an employee under the Collector of Revenue at Delhi and had six sons of whom Ramchundra was one. Ramchundra had his primary education for some time in private schools and then joined the English Government School at Delhi where he studied for six years with government scholarships. He created for himself a taste for mathematics by self-study during those years. This school which came under the then Bengal Presidency was upgraded to a college in 1841. Ramchundra won a senior scholarship of Rs.30/- p.m. He was appointed teacher of European science in the Oriental Department of the college in 1844 with an additional compensation of Rs.50/- p.m. for teaching in the vernacular. A vernacular translation society was formed and Ramchundra translated into Urdu books on algebra, trigonometry and calculus under the aegis of the society. Modern science education challenged the traditional old beliefs of the Hindus and Muslims and this created some resistance to the initiation of youngsters in modern science. The attempt of Ramchundra to run a magazine to spread modern science was scorned at by the traditionalists. He was branded as tending to conversion to Christianity which he challenged saying that no Christian can convert him. At that time Ramchundra held the belief that religion and philosophy are personal and not an outcome of bookish learning. Ramchundra was encouraged in his attempts to get over traditional belief to spread modern science by the Britishers who were in the administration at Delhi.

In 1850, Ramchundra published on his own the first edition of his "Problems of Maxima and Minima". He was expecting good response for his work and appreciation and awards on this account. Quite contrary to his expectation the book was badly reviewed in the Calcutta Review. He wrote a counter to the review and got it published in the Englishman. The native people were unhappy with his writing a book in English. He had a consolation in the form of one D.Bethune, a member of the Supreme Council, getting from him 36 copies for a donation of Rs.200/- and sending a few copies to England. Later in 1858-59, Ramchundra was sanctioned a dress of honour and an award of Rs.2000/-.

Ramchundra, who belonged to a Kayastha Hindu family, took faith in Christianity, but, however, baptized himself in May 1852, after a long hesitation. Ramchundra claimed to have studied the Koran and the Bhagavadgita in English translation and engaged himself in conversations with scholars knowing these books in original before concluding that Jesus Christ was the only Saviour. He was targeted by the mutineers against the English during June 1857 and was saved by a Brahmin Zamindar and a Jat servant, as he admitted.

Augustus De Morgan (popularly known for his set-theoretic laws), who received one of the copies of Ramchundra's book published in 1850 sent to England by D.Bethune was impressed with his book. It was on De Morgan's recommendation that the honours and award mentioned earlier were made to Ramchundra by the British rulers of India. De Morgan got the book reprinted in England in 1859 with a view to get it noticed by European scholars. De Morgan held the views that the Hindus who were responsible for initiating studies in arithmetic and algebra and whose knowledge was

spread in Europe, for which the west should be indebted to them, lost their originality (which De Morgan called speculation) and got rules framed by their predecessors by heart for applications. De Morgan's appreciation for Ramchundra is in terms of his original thinking in removing the need of calculus in solving problems in extrema. De Morgan, who had Indian connections (see the article in the Mathematics Teacher Vol.41, Issue 1 & 2, pp. 110 to 112 by R.C.Gupta), was unfortunately unaware of work relating to the initial stages of analysis and the discovery of the series for sine, cosine, arctan some two centuries earlier to Newton by Keralite mathematicians exposed by the British Civil servant C.M. Whisk in 1835. This would have made him refrain from the remark that original thinking was lost in the Hindus after Āryabhata, Bhaskara and Brahmagupta, to some of whose work De Morgan had access through English translations. In all likelihood, the British rulers, in general, did not encourage or appreciate originality in the Hindus. There were personal exceptions to this situation as illustrated by Whish, Bethune and the like. Most of the Indians, like Ramchundra, who got the benefit of English education in those days, who did not get introduced to traditional learning by the Hindus or Muslims, could only appreciate western culture and knowledge to the extent of expressing scorn for their native knowledge with no attempt to set the misconceptions, if found therein, right and carrying forward this knowledge with their originality. It is also very likely that Ramchundra would not get the awards and his book printed in England, if he did not become Yesudas Ramchundra!

Ramchundra published his second book titled: New method of differential calculus in 1861. He served as tutor to the Maharaja of Patiala and was honoured by him through

the award of a jagir. He was appointed Director of Public Instruction of that state and did his best for the improvement of education in the state for over a decade.

Ramchundra was a philanthropically inclined person till the very end of his life. His end came on 11 August 1880 after he suffered from a paralysis of his limbs. Ramchundra's was a life of lack of recognition by his countrymen as well as the westerners. His book should have been, otherwise, as popular, if not more, than books relating to extrema problems published in the New Mathematics Series and reprinted by the Mathematical Association of America as late as 1981.

The following information was provided by Dr.R.C.Gupta about Ramchundra (spelt as Ramachandra or Ramachundra by writers other than De Morgan): He was married to Sita being in her teens, impaired in hearing and speech but being, however, a daughter of a very rich man Kushal Rai. Besides translating some text books in mathematics written in English by Englishmen as mentioned earlier, Ramchundra wrote in Urdu Sari-ul-Fahm a primer in mathematics, Asool-i-Jabr-o-Mugabal (Principles of Algebra) and Asool-i-Ilm-i-Hisab-Juziat-o-Kuliyat (Principles of Maxima and Minima) published around 1849 from Delhi. Towards the end of his life he wrote on religious topics: Aitaraz-i-Quran (Delhi, 1875), a criticism of Islam; Bidat Ipai Mazhab (Delhi, 1880), about sins in Christianity.

CONTENTS

Page

Between Us ..... *iii*

Editor's Introduction ..... *iv*

About the Author ..... *vi*

INTRODUCTION 1

CHAPTER-1 10

Problems which involve Extrema of Linear or Quadratic Polynomials

CHAPTER-2 71

Problems in Maximum and Minimum in the Solution of which Cubic Equations are used

CHAPTER-3 104

Problems of Maxima, Minima in the Solutions of which Equations of Fourth to Seventh Degree are used

CHAPTER-4 132

Problems of maxima and minima in which two or more variable quantities are used

Supplement 158

Additional Problems 163

INTRODUCTION

The original introduction by the author was, perhaps, intended to make the booklet self-contained. It starts with explaining the concept of equation, of a root of an equation, a proof of the remainder or factor theorem etc. We make a selection of the author's writing in the original which many of the readers may not be familiar with, indicating those items which are not reproduced here.

In (2) of the introduction the equation of the parabola in the form $y^2 = 4ax$ is derived using the focus directrix property. In (3) is derived the standard equation of the ellipse defining it as the set of all points the sum of whose distances from two fixed points (foci) is constant.

In (4) are defined

(i) an ellipsoid as a solid (surface) whose sections by its thin axial planes is an ellipse.

(ii) a spheroid as a surface of revolution of an ellipse about one of its two axes.

(iii) a sphere as a surface of revolution of a circle about one of its diameters.

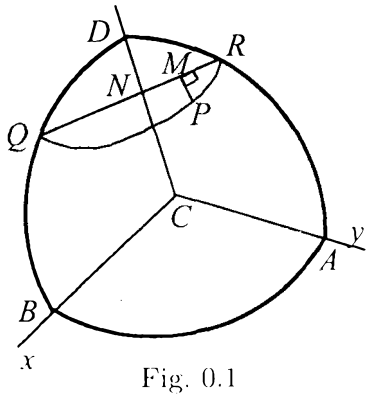


Fig. 0.1

Fig.0.1. shows one eighth of an ellipsoid in the positive octant. AB shows the section of the surface by the xy plane which is part of an ellipse, center C (origin). The surface (part of it) can be traced by lifting this ellipse parallel to itself and reducing its size. e.g. N is one position of C in this process QR is the position of the ellipse. P is a variable point in the plane of QR such that $CN = z$. Let M be the foot of the perpendicular from P on NR and let $PN = y$, $NM = x$. Let $CA = a$, $CB = b$, $CD = c$ (the semiaxes of the ellipsoid). Let $NR = x_1$, $NQ = y_1$. Then, from ellipse QPR,

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1.$$

From ellipses DQB, DRA,

$$\frac{x_1^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \frac{y_1^2}{b^2} + \frac{z^2}{c^2} = 1$$

so that $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$. Using this in the equation

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

by means of multiplication by $\frac{x_1^2}{a^2}\left(= \frac{y_1^2}{b^2}\right)$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 - \frac{z^2}{c^2}$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

yielding the equation of the ellipsoid.

When $a = b$, the equation of the spheroid is obtained

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

For the sphere $a = b = c$ (= the radius) so that we have the equation

$$x^2 + y^2 + z^2 = a^2.$$

In (5) the derivation for the formula for the area of a triangle in the forms $\frac{1}{2}bh$, $\sqrt{s(s-a)(s-b)(s-c)}$ where b is the base of the triangle, h is the altitude to the base, a, c are the other sides, $s = \dfrac{a+b+c}{2}$, the semiperimeter, is derived.

In (6) is explained the genesis of the formula πd for the circumference of a circle of diameter d. As in Fig.0.2 let AB, BC be equal arcs of a circle center E. Draw the diameter DA and produce it to F such that chord $BD = BF$, consider Δs DEB, DBF which are isosceles. Since $\angle D$ is common, these are similar (equiangular) triangles. Thus $DE : DB = DB : DF$.

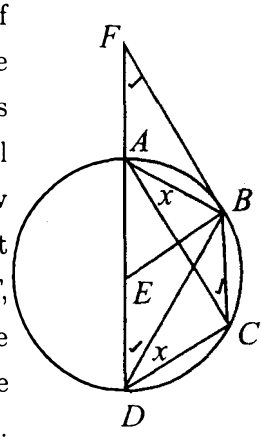


Fig. 0.2

Now $\Delta AFB \equiv \Delta DCB$. For $\angle BDC = \angle ADB$ (angles subtended by equal arcs at the circumference) $= \angle AFB$ (since ΔFBD is isosceles). By property of cyclic quadrilateral $\angle FAB = \angle BCD$. Also $BF = BD$ by construction. Congruence therefore follows. Thus $DF = DA + AF = 2DE + AF = 2DE + DC$.

Hence

$$DE : DB = DB : DF$$

$$\Rightarrow \quad DE : DB = DB : 2DE + DC$$

so that $DB^2 = 2DE^2 + DE \cdot DC$. If the circle is of unit radius, then $DE = 1$ and so $DB = \sqrt{2 + DC}$. Calling the arc

which remains in a semicircle after omitting the given arc as supplemental arc of the given arc, what we have so far shown is that

Chord of supplemental arc

$$= \sqrt{2 + \text{chord of supplemental arc of twice the given arc}}$$

or equivalently

$$\sqrt{2 + \text{ chord of supplemental arc of an arc}}$$

= chord of supplemental arc of half of the given arc.

If A, C are two adjacent vertices of a regular hexagon inscribed in the unit circle, then the side of the hexagon, AC is again one unit and so $DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1} = \sqrt{3}$; But DC is the supplemental chord of $\frac{1}{6}$th of the circumference of the circle. If follows that the chord of $\frac{1}{12}$ of the circumference is $\sqrt{2 + \sqrt{3}}$, and of $\frac{1}{24}$ of the circumference is $\sqrt{2 + \sqrt{2 + \sqrt{3}}}$. Proceeding thus the chord of $\frac{1}{1536}$ of the circumference is found to have square almost equal to 3.9999832669. Thus the difference between the square of the diameter and this quantity is

$$4 - 3.9999832669 = 0.0000167331$$

This is approximately the square of the chord corresponding to $\frac{1}{1536}$ of the circumference. Thus, approximately

$$\begin{aligned} &\frac{1}{1536} \text{ of the circumference} \\ &\qquad = \sqrt{0.0000167331} \\ &\qquad = 0.0040906112. \end{aligned}$$

Thus the circumference is approximately $1536 \times 0.0040906112 = 6.2831788$ for a circle of diameter 2 units. Thus the circumference of the unit circle is $3.1415894 = 3.14159$ approx.

In (7) the author explains how the area of any circle is equal to the area of a rectangle with sides equal to $\frac{1}{2}$ of circumference and $\frac{1}{2}$ of diameter. As in Fig 0.3. Consider a regular hexagon inscribed in the circle. By joining the vertices to the center of the circle we get 6 equal isosceles triangles. Let $CD \perp AB$ one of the sides. Each triangle has area = area of a rectangle with sides equal to CD and half of the side.

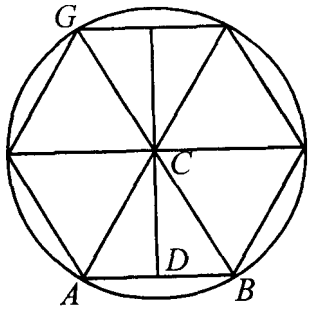


Fig. 0.3

Thus, area of the hexagon = area of a rectangle whose sides are the common height of the triangle and half of the perimeter of the hexagon. If we make the regular polygon have more and more sides the common height of the triangle so obtained is almost equal to the radius of the circle and the perimeter is almost equal to the circumference of the circle. Thus, in the limit area of the circle $= \frac{1}{2}$ of diameter $\times \frac{1}{2}$ of circumference.

In (8) it is shown that every sphere is two thirds in volume of its circumscribing cylinder. $ABCD$ is the circumscribing cylinder of the sphere centered at I (see Fig 0.4).

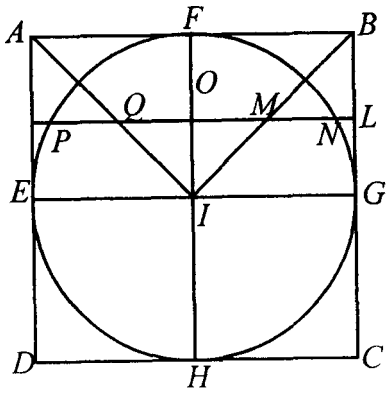


Fig. 0.4

Let OL be a cross section of the cylinder parallel to its base making a section QM on the cone AIB half way between EG and AB. It is known that volume of cone AIB : volume of cone $QIM = IF^3 : IK^3$ (where F, K are centers of the bases of the cones) $= FH^3 : (FH - 2FK)^3$. Volume of cone AIB : frustum (i.e., bottom part of AIB) $ABMQ$

$$= FH^3 : FH^3 - (FH - 2FK)^3$$
$$= FH^3 : 6FH^2 \cdot FK - 12FH \cdot FK^2 + 8FK^3$$

But cone $AIB =$ one third of cylinder $ABGE$. So, cylinder AG : frustum $ABMQ = 3FH^3 : 6FH^2.FK - 12FH \cdot FK^2 + 8FK^3$.

Also, cylinder AL : cylinder $AG = FK : FI$. So,
cylinder $ABGE$: frustum $ABMQ = 6FH^2 :$

$$6FH^2 - 12FH \cdot FK + 8FK^2 . \qquad (*)$$

Now, $IK = KM$ and so $IK^2 + KN^2 = KM^2 + KN^2 = IN^2 = IG^2 = KL^2$. So area of circle with radius $KL =$ area of circle with radius $KM +$ area of circle with radius KN. i.e., the area of the section of the cylinder with center K and radius $KL =$ sum of the areas of the sections by the same plane of the cone and the sphere. Thus adding up these sections we have

Volume of the cylindrical part above $EG =$ Volume of the cone $AIB +$ Volume of the hemisphere EFG.

By $(*)$ cylinder AL : Part of the sphere in PFN

$$= 6FH^2 : 12FH \cdot FK - 8FK^2$$
$$= \frac{3}{2}FH^2 : FK(3FH - 2FK)$$

But volume of cylinder $AL =$ area of circular base of diameter $AB \times$ height $FK =$ circle $EFGH \times FK$.

Hence segment $PFN = \frac{2}{3} \frac{\text{circle } EFGH}{FH^2} (3FH - 2FK)FK^2$. When $FK = FH$, then the sphere $= \frac{2}{3}$ cylinder noting that cylinder $AL =$ frustrum $ABMQ =$ segment PFN.

In (9) the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is derived. The equation of the circle on major axis AP as diameter is $y^2 = a^2 - x^2$. If the ordinate of two points on the same ordinate on the ellipse and the circle are y and y^1, then $y : y^1 = 2b : 2a$.

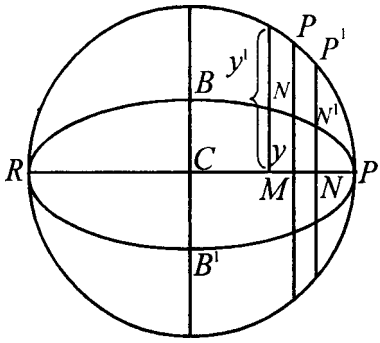


Fig. 0.5

Consider ordinates MP, NP^1 erected closely forming a strip. Let us denote CM by x, MN by y and $MP = y^1$. The strip cut by the ellipse: The strip cut by the circle $= y : y^1 = b : a$. Thus by adding such strips area of the ellipse: area of the circle $= b : a$

$$\text{Hence area of the ellipse} = \frac{\text{area of the circle} \times b}{a}$$

$$= \frac{\pi a^2 b}{a} = \pi ab.$$

In (10) a method of deriving the formula for the sum of the squares of the first n natural numbers is given.

Let $1 + 4 + 9 + \cdots + n^2 = Pn^3 + Qn^2 + Rn + S$. Putting $n = 1, 2, 3, 4$ we have

$$\begin{aligned} P + Q + R + S &= 1 \\ 8P + 4Q + 2R + S &= 5 \\ 27P + 9Q + 3R + S &= 14 \\ 64P + 16q + 4R + S &= 30 \end{aligned}$$

The solution of this linear system gives the values of P, Q, R, S as $P = \frac{1}{3}$, $Q = \frac{1}{2}$, $R = \frac{1}{6}$ and $S = 0$. Thus

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{n}{6} = \frac{n}{6}(2n^2 + 3n + 1)$$
$$= \frac{n}{6}(n+1)(2n+1)$$

Note: The author, though inexactly stated, observes that $\sum_{k=1}^{n} k^2$ is asymptotic to $\frac{n^3}{3}$ as $n \to \infty$.

In (11) the author finds the area enclosed by the parabola $y^2 = 4ax$ bounded by the axis and an ordinate (see Fig.0.6).

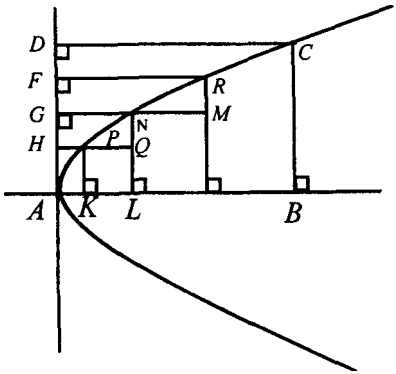


Fig. 0.6

$KP^2 = 4a.AK$, i.e., $AH^2 = 4aHP$.

i.e., $HP = \frac{AH^2}{4a}$; $LN^2 = 4a \cdot AL$.

So $AG^2 = 4aGN$ or $GN = \frac{AG^2}{4a}$

$AF^2 = 4aFR$ or $FR = \dfrac{AF^2}{4a}$.

Let $AH = HG = GF = FI = \ldots = \dfrac{AD}{n}$.

Rectangle $HPKA = HP \times AH = \dfrac{AH^3}{4a} = \dfrac{AD^3}{4an^3}$.

Rectangle $HGNQ = HG \times GN = HG \times \dfrac{AG^2}{4a} = \dfrac{4AD^3}{4an^3}$.

Rectangle $GFRM = GF \times FR = GF \times \dfrac{AF^2}{4a} = \dfrac{9AD^3}{4an^3}$

Sum of these rectangles $= \dfrac{AD^3}{4an^3} + \dfrac{4AD^3}{4an^3} + \dfrac{9AD^3}{4an^3} + \cdots$

$= \dfrac{AD^3}{4an^3}(1 + 4 + 9 + \cdots + n^2) = \dfrac{AD^3}{4an^3}\, n\dfrac{(n+1)(2n+1)}{6}$

$= \dfrac{AD^2}{4a} \times \dfrac{AD}{n^3}\dfrac{n(n+1)(2n+1)}{6} = DC \times \dfrac{AD}{n^2}\dfrac{(n+1)}{2}\dfrac{(2n+1)}{3}$.

Making the number n of parts into which the area is divided larger and larger the area $\Delta PNRCD$ outside the parabola is given approximately by $n = \dfrac{DC \times AD}{3}$.

The area of the parabolic part is therefore rectangle $ABCD$ –

area $APNRCD = DC \times AD - \dfrac{DC \times AD}{3} = \dfrac{2DC \times AD}{3}$.

Note: The author uses the first principles of integration in this derivation and assumes that AD has been divided in n equal parts where n is eventually made to tend to infinity.

CHAPTER I

Problems which involve Extrema of Linear or Quadratic Polynomials

Problem 1. Given a positive real number a to find $x, y > 0$ such that $x + y = a$ and xy is maximum.

Solution. Let us write $y = a - x$ and assume that r is the maximum value of $xy = x(a - x)$ for a particular choice of x. i.e., $x(a - x) = r$, i.e., $ax - x^2 = r$, or $x^2 - ax + r = 0$, i.e., $(x - \frac{a}{2})^2 + r - \frac{a^2}{4} = 0$. If $r > \frac{a^2}{4}$, then the above equality cannot hold for a real value of x. Again, the value $\frac{a^2}{4}$ is assumed for $x = \frac{a}{2}$ by $x(a - x)$. Thus the maximum value r of $x(a - x)$ is $\frac{a^2}{4}$ and it is assumed when $x = \frac{a}{2}$.

Remark. Instead of arguing that the maximum value cannot be larger than $\frac{a^2}{4}$, assume that the value is assumed for $x = \frac{a}{2} + y$ for some real y. Then $ax - x^2 = ay + \frac{a^2}{2} - y^2 - ay - \frac{a^2}{4} = \frac{a^2}{4} - y^2$ which attains its maximum when $y = 0$.

Problem 2. To determine the rectangle of the greatest area, inscribed (i.e., with vertice of the rectangle lying on the sides of the triangle) in a given triangle.

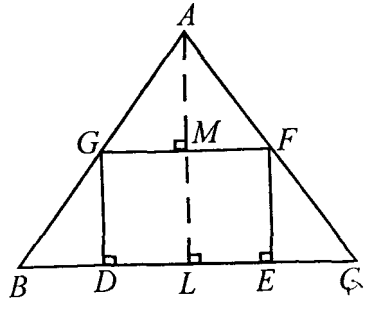


Fig. 1.1.

Solution. Let ABC be the given triangle where we assume that B and C are acute angles,

and $DEFG$ a rectangle inscribed therein as shown in the Fig. 1.1. (Note that two of the vertices of the rectangle should lie on the same side of the triangle).

Let AL be an altitude of the triangle meeting GF at M. Then, Δs AGF and ABC being similar,

$$\frac{AM}{AL} = \frac{GF}{BC}.$$

Thus, if $GD = x$, $AL = d$, $BC = a$, then

$$\frac{d-x}{d} = \frac{GF}{a} \text{ or } GF = \frac{ad - ax}{d}$$

So the area of $DEFG$ is $GF \times GD = \frac{ad-ax}{d} \times x = \frac{adx - ax^2}{d} = \frac{a}{d}(dx - x^2)$. Here a, d are given constants and so the problem reduces to finding the maximum of $dx - x^2$. As in the solution of Problem 1, this maximum is $\frac{d^2}{4}$ and the corresponding value of x is $\frac{d}{2}$. In other words, M is the mid-point of AL. This solves the problem if the rectangle has one of its sides along the side BC of ΔABC. It follows that the largest rectangle is obtained by choosing the largest of the sides of ΔABC to have one of the sides of the rectangle lying along it.

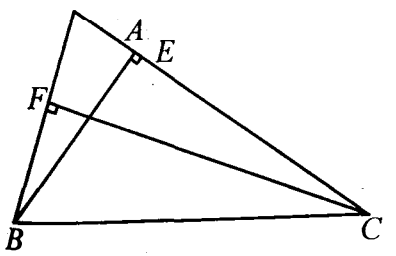


Fig.1.2

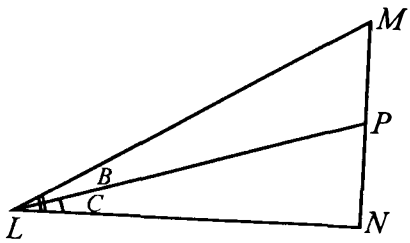


Fig.1.3

Note: If in ΔABC, $\angle B > \angle C$, then the altitude from B is smaller than the one from C. For, if BE, CF are altitudes (see Fig. 1.2), then, ΔBCF and ΔCBE are congruent to ΔLMN and ΔLPN respectively in Fig.1.3 so that $BE = PN < MN = CF$.

Problem 3. To find the right angled triangle with given hypotenuse which has maximum area .

Solution. Let ΔABC be right angled at B and let the hypotenuse $AC = a$, a constant (see Fig.1.4). Let $AB = x$, $BC = y$. The problem is: given $x^2 + y^2 = a^2$ to find maximum of $\frac{1}{2}xy$. Now, $y = \sqrt{a^2 - x^2}$ and hence $xy = x\sqrt{a^2 - x^2}$; we are to find the maximum of $x\sqrt{a^2 - x^2}$ or, equivalently, of $x^2(a^2 - x^2) = a^2x^2 - x^4$. i.e., to find the maximum of $a^2X - X^2$, where $X = x^2$. From the solution of Problem 1, again, the maximum value is $\frac{a^4}{4}$ so that the maximum value of $\frac{1}{2}xy$ is $\frac{a^2}{4}$ and this value is assumed when $x^2 = X = \frac{a^2}{2}$, i.e., $x = \frac{a}{\sqrt{2}}$. Then $y = \sqrt{a^2 - x^2} = \frac{a}{\sqrt{2}}$. In other words, the right angled triangle is isosceles.

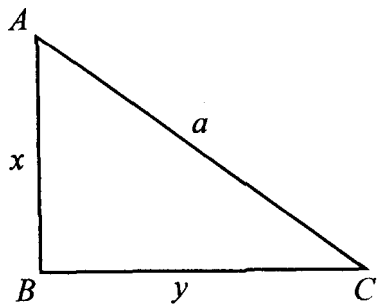


Fig. 1.4

Problem 4. To find the minimum of the sum of the sides containing the right angle of a right angled triangle with prescribed area.

Solution. See Fig.1.4. Given $\frac{1}{2}xy = a$ or $xy = 2a$, a a constant, to find the maximum of $x + y$. Now, $y = \frac{2a}{x}$ and, so, $x + y = x + \frac{2a}{x} = \frac{x^2 + 2a}{x}$. If the minimum is r, then the corresponding value of x is such that $\frac{x^2 + 2a}{x} = r$, i.e., $x^2 - rx + 2a = 0$, i.e., $(x - \frac{r}{2})^2 + 2a - \frac{r^2}{4} = 0$. Hence $2a - \frac{r^2}{4} \not> 0$, or, $\frac{r^2}{4} \not< 2a$, i.e., $r \not< 2\sqrt{2a}$. If $x = \sqrt{2a} = y$, then $xy = 2a$ and $x + y = 2\sqrt{2a}$. Hence the minimum value of $x + y$, viz., $r = 2\sqrt{2a}$ and it is attained when the right triangle is isosceles.

Problem 5. Given a line segment, to divide it into two segments such that the sum of the areas of the squares on these two segments is minimum.

Solution. If a is the length of the given line segment and x is the length of one of the two segments cut there from, then the other segment is of length $a - x$. The problem is to find the minimum of $x^2 + (a-x)^2 = 2x^2 - 2ax + a^2$. It suffices to find the minimum of $2x^2 - 2ax$ or of $x^2 - ax$ since a is a constant. Now, if $x^2 - ax = r$, then $(x - \frac{a}{2})^2 - \frac{a^2}{4} - r = 0$. Hence $\frac{a^2}{4} + r \not< 0$, i.e., $r \not< -\frac{a^2}{4}$. If $x = \frac{a}{2}$, then $x^2 - ax = \frac{a^2}{4} - \frac{a^2}{2} = -\frac{a^2}{4}$ so that the minimum value of $x^2 - ax$ is $-\frac{a^2}{4}$ and the minimum value of $x^2 + (a - x)^2$ is $-2\frac{a^2}{4} + a^2 = \frac{a^2}{2}$. Thus the minimum is obtained by bisecting the given line segment.

Note: To find the minimum of $x^2 - ax$ is equivalent to finding maximum of $ax - x^2$ since the minimum value of $x^2 - ax$ is

the negative of the maximum of $ax - x^2$. Thus the solution of Problem 5 reduces to the solution of Problem 1.

Problem 8. To find the solid right cone of circular maximum volume with prescribed surface area.

Solution. Let the radius of the base of the cone be r and its slant height l (see Fig.1.5). If s is the prescribed surface area $s = \pi r^2 + \pi r l$. Thus $l = \frac{s}{\pi r} - r$. Hence the (vertical) height of the cone

$$\begin{aligned} h &= \sqrt{l^2 - r^2} \\ &= \sqrt{(\frac{s}{\pi r} - r)^2 - r^2} \\ &= \sqrt{\frac{s^2}{\pi^2 r^2} - \frac{2s}{\pi}} \end{aligned}$$

So, the volume of the cone

$$\begin{aligned} v &= \frac{1}{3}\pi r^2 h \\ &= \frac{\pi r^2}{3}\sqrt{\frac{s^2}{\pi^2 r^2} - \frac{2s}{\pi}} \end{aligned}$$

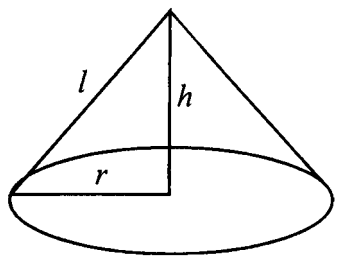


Fig. 1.5

v is maximum when $r^2(s^2 - 2\pi s r^2) = s(sr^2 - 2\pi r^4)$ is maximum. s being constant, v is maximum when $\frac{sr^2}{2\pi} - r^4$ is maximum. Putting $a = \frac{s}{2\pi}$ and $r^2 = x$, v is maximum when $ax - x^2 = x(a - x)$ is maximum. From the solution to Problem 1, the maximum value of $x(a - x)$ is reached when $x = \frac{a}{2}$ and is $\frac{a^2}{4}$. Thus, the maximum value of v^2 is $\frac{4\pi^2}{9}\frac{s^4}{4\pi^2} \cdot \frac{1}{4}$ so that the maximum value of v is $\frac{s^2}{6}$ and this is reached when

$r^2 = \frac{a}{2} = \frac{s}{4\pi} = \frac{\pi r^2 + \pi rl}{4\pi} = \frac{r^2}{4} + \frac{rl}{4}$, i.e., $\frac{3r^2}{4} = \frac{rl}{4}$ or $\frac{l}{r} = 3$, i.e., when the slant height is three times the radius of the base. This is equivalent to $\frac{h^2}{(2r)^2} = 2$.

Note: The solution to the dual problem of finding the solid right circular cone with least surface area and with prescribed volume is the same.

Problem 7. Given two rays to find a straight line passing through a given point not on the rays which makes intercepts on the rays whose sum is the least.

Solution. Let AR, AS be the rays and P the given point. Draw PD, PE respectively parallel to AS and AR (see Fig.1.6) meeting AR, AS at D, E respectively. Let $PD = a$, $PE = b$ and note that a, b are constants. Let the variable line through P meet AR, AS respectively ar L and M. Let $LD = x$.

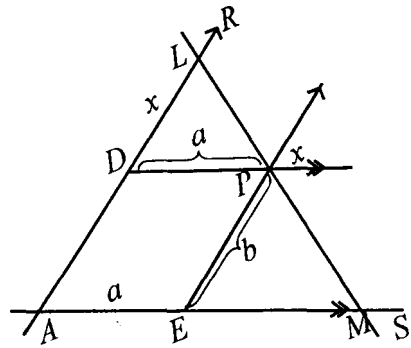


Fig. 1.6

Then from Δs LDP, PEM which are similar $\frac{LD}{PE} = \frac{DP}{EM}$, i.e., $\frac{x}{b} = \frac{a}{EM}$ or $EM = \frac{ab}{x}$. Hence $AL + AM = b + x + a + EM = a + b + x + \frac{ab}{x}$ is to be minimized. $b + a$ being constant, it suffices to minimize $x + \frac{ab}{x}$. If $x + \frac{ab}{x} = r$, then $x^2 - rx + ab = 0$.

As earlier (see the solution to Problem 4, for instance), the minimum value of $x + \frac{ab}{x}$ is r where $\frac{r^2}{4} = ab$, i.e., $r^2 = 4ab$, i.e., $r = 2\sqrt{ab}$ and this value is reached when $x = \sqrt{ab}$. The minimum value of $AL + AM$ is therefore

$$a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2.$$

Remark. Draw Fig 1.6 for the cases when P lies in the other three regions formed by the rays AR, AS and check up whether the same solution holds.

Problem 8. Two particles move with uniform velocities in a given ratio from two points starting at the same time in two given directions from the two given points. To find the position where they are at the least distance from each other.

Solution. The given points are A, B and the given directions are AP, BQ (see Fig.1.7). Let AP, BQ meet at C. M, N are the positions of the two particles on AP, BQ respectively at a particular time. Let the velocity of the particle moving on BQ bear to that of the particle moving on AP the ratio $n : m$. Let $CN = x$. Draw perpendiculars to AP from B and N meeting it at D, E respectively. Let $CA = a$, $CB = b$, $CD = c$, which are

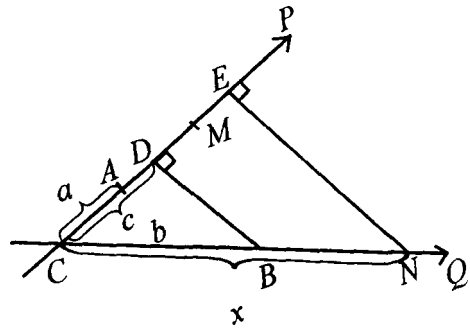


Fig. 1.7

constant Δs CBD, CNE are similar so that

$$\frac{CB}{CN} = \frac{CD}{CE}, \qquad \text{i.e.,} \quad \frac{b}{x} = \frac{c}{CE}$$

or $CE = \frac{cx}{b}$. Also, by assumption, about the velocities of the two particles $\frac{BN}{AM} = \frac{n}{m} = \frac{x-b}{AM}$ or $AM = \frac{m(x-b)}{n}$.

Hence

$$CM = CA + AM = a + \frac{m(x-b)}{n} = \frac{na - mb}{n} + \frac{m}{n}x,$$
$$= d + \frac{m}{n}x, \, say.$$

By the cosine formula applied to ΔCNM

$$\begin{aligned} MN^2 &= CN^2 + CM^2 - 2CN.CM.\cos NCM \\ &= CN^2 + CM^2 - 2CM.CE \\ &= \left(d + \frac{m}{n}x\right)^2 + x^2 - 2\left(d + \frac{m}{n}x\right)\frac{cx}{b} \\ &= d^2 + \frac{m^2}{n^2}x^2 + 2\frac{m}{n}dx + x^2 - 2\frac{cd}{b}x - \frac{2mc}{bn}x^2 \\ &= \left(\frac{m^2}{n^2} - \frac{2mc}{nb} + 1\right)x^2 - \left(\frac{2cd}{b} - \frac{2m}{n}d\right)x + d^2 \\ &= A(x^2 - Bx + C), \end{aligned}$$

where

$$A = \frac{m^2}{n^2} - \frac{2mc}{nb} + 1 = \frac{m^2b - 2mnc + n^2b}{n^2b}, \quad B = \frac{2d(nc - mb)}{Anb}$$

and $C = \frac{d^2}{A}$. Now, $A > 0$ since discriminant of $m^2b - 2mnc + n^2b$ as a homogeneous quadratic in m, n, $4c^2 - 4b^2 < 0$ since $b > c$. Thus the problem reduces to one of finding the minimum

value of $x^2 - Bx$ which is $-\frac{B^2}{4}$ and is attained for $x = \frac{B}{2}$ (see solution to Problem 5). Thus, the minimum value of MN^2 is

$$A\left(\frac{B^2}{4} - B.\frac{B}{2} + C\right) = A\left(C - \frac{B^2}{4}\right)$$

Now, $\frac{B^2}{4} = \frac{d^2}{A^2}\frac{(nc - mb)^2}{n^2b^2}$ while $C = \frac{d^2}{A}$. Thus

$$\frac{C}{B^2/4} = \frac{An^2b^2}{(nc - mb)^2} = \frac{(m^2b^2 - 2mnbc + n^2b^2)}{m^2b^2 - 2mnbc + n^2c^2} > 1$$

since $b > c$. Thus $C - \frac{B^2}{4} > 0$ and MN has a positive minimum value when $x = \frac{B}{2}$. AM, AN are easily calculated.

Problem 9. The particle M moves with uniform velocity m in the direction AQ starting from A. Another particle N moves from rest at B with uniform velocity n. To determine the direction from B in which N lies on AQ.

Solution. As in Fig.1.8, let N be the position of the particle which started from B when it lies on AQ.

Let M be the position of the first particle at this moment of time. Let C be the foot of the perpendicular from B on AQ, $AC = a$, $BC = b$, both being constants.

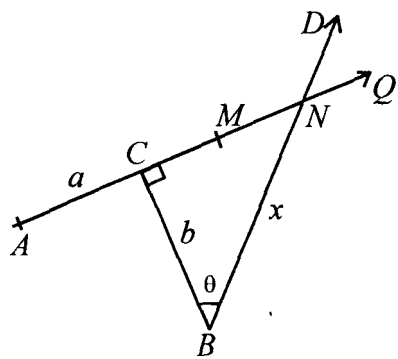


Fig. 1.8

Let $BN = x$. From the given details $\frac{n}{m} = \frac{BN}{AM} = \frac{x}{AM}$, so that $AM = \frac{mx}{n}$. $CM = AM - AC = \frac{mx}{n} - a$, so $MN = CN - CM = -\frac{mx}{n} + a + \sqrt{x^2 - b^2}$.

Write $r = -\frac{mx}{n} + a + \sqrt{x^2 - b^2}$ so that we are interested in the maximum of r. Now, $\sqrt{x^2 - b^2} = \frac{mx}{n} - a + r$, i.e.,

$$x^2 - b^2 = \frac{m^2x^2}{n^2} + a^2 + r^2 - \frac{2max}{n} - 2ar + \frac{2mxr}{n}$$

or

$$x^2(1 - \frac{m^2}{n^2}) + \frac{2m}{n}(a - r)x - b^2 = a^2 + r^2 - 2ar$$

$$= (a - r)^2$$

$$x^2\frac{n^2 - m^2}{n^2} + \frac{2m}{n}(a - r)x = (a - r)^2 + b^2$$

$$x^2 - \frac{2mn}{m^2 - n^2}(a - r)x = -\frac{n^2}{m^2 - n^2}[(a - r)^2 + b^2]$$

$$\{x - \tfrac{mn}{m^2 - n^2}(a - r)\}^2$$

$$= \frac{m^2n^2(a - r)^2}{(m^2 - n^2)^2} - \frac{n^2}{m^2 - n^2}[(a - r)^2 + b^2]$$

$$= \frac{m^2n^2(a - r)^2}{(m^2 - n^2)^2} - \frac{n^2(m^2 - n^2)}{(m^2 - n^2)^2}[(a - r)^2 + b^2]$$

$$= \frac{m^2n^2(a - r)^2 - m^2n^2(a - r)^2 + n^4(a - r)^2}{(m^2 - n^2)^2}$$

$$- \frac{n^2(m^2 - n^2)b^2}{(m^2 - n^2)^2}$$

$$= \frac{n^4(a - r)^2 - n^2(m^2 - n^2)b^2}{(m^2 - n^2)^2}$$

Note that the l.h.s. is always greater than or equal to zero so that $n^4(a-r)^2 \geq n^2b^2(m^2-n^2)$ or $n^2(a-r)^2 \geq b^2(m^2-n^2)$. Hence, when $m \geq n$,

$$\begin{aligned} (a-r) &\geq \frac{b}{n}\sqrt{m^2-n^2} \\ \text{or} \qquad r &\leq a - \frac{b}{n}\sqrt{m^2-n^2}. \end{aligned}$$

Thus, the maximum value of r is $a - \frac{b}{n}\sqrt{m^2-n^2}$, it being assumed that $a \geq \frac{b}{n}\sqrt{m^2-n^2}$.

$$\begin{aligned} \text{That is} \qquad a^2n^2 &\geq b^2(m^2-n^2) \\ \text{or} \qquad (a^2+b^2)n^2 &\geq b^2m^2 \\ \text{or} \qquad \frac{m^2}{n^2} &\leq \frac{a^2+b^2}{b^2} \\ \text{or} \qquad \frac{m}{n} &\leq \frac{\sqrt{a^2+b^2}}{b}. \end{aligned}$$

Remarks. 1. If $m < n$, there is no maximum value for MN in Fig. 1.8. If $C\hat{B}N = \theta$, then, at time t when position of M and N are as shown in the figure,

$$\begin{aligned} MN &= -mt + a + \sqrt{(nt)^2 - b^2} \\ &= -\frac{m}{n}(nt) + a + \sqrt{n^2t^2 - b^2} \\ &= -\frac{m}{n}\frac{b}{\cos\theta} + a + \sqrt{\frac{b^2}{\cos^2\theta} - b^2} \\ &= -\frac{m}{n}\frac{b}{\cos\theta} + a + \frac{b}{\cos\theta}\sin\theta \\ &= +\frac{b}{\cos\theta}(\sin\theta - \frac{m}{n}) + a \end{aligned}$$

so that $\sin\theta$ is larger than $\frac{m}{n}$ for θ between θ_0 and $\frac{\pi}{2}$ for some $\theta_0 > 0$ (Note that $\sin\theta_0 = \frac{m}{n}$). Thus as $\theta \to \frac{\pi}{2}$, $MN \to \infty$ so that there is no maximum value for MN.

2. If M lies beyond N on AQ, then in the notation of Remark 1 above,

$$\begin{aligned} NM = CM - CN &= \frac{mx}{n} - a - \sqrt{x^2 - b^2} \\ &= \frac{b}{\cos\theta}\left(\frac{m}{n} - \sin\theta\right) - a \end{aligned}$$

so that, if $m > n$, making $\theta \to \frac{\pi}{2}$, $NM \to \infty$. Hence, there is no maximum for NM in this case. If $m < n$, after a stage for θ, N crosses M to the right and MN becomes larger and larger as θ increases to $\pi/2$. So, there is no maximum for MN in this case too.

3. If A lies to the right of C as in Fig. 1.9 and $m > n$, in the earlier notation

$$\begin{aligned} MN &= \frac{b}{\cos\theta}\sin\theta - \frac{m}{n}\frac{b}{\cos\theta} - a \\ &= \frac{b}{\cos\theta}\left(\sin\theta - \frac{m}{n}\right) - a \end{aligned}$$

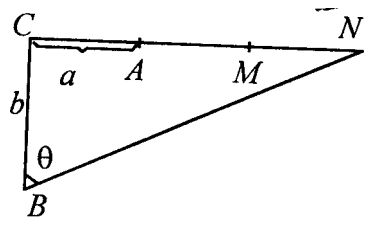


Fig. 1.9

so that it is impossible to have M lying to the left of N and when it lies to the right of N

$$\begin{aligned} NM &= \frac{m}{n}\frac{b}{\cos\theta} + a - \frac{b}{\cos\theta}\sin\theta \\ &= \frac{b}{\cos\theta}\left(\frac{m}{n} - \sin\theta\right) + a \end{aligned}$$

so that $NM \to \infty$ as $\theta \to \frac{\pi}{2}$, there being no maximum for NM.

4. If $m = n$

$$MN = \frac{b}{\cos\theta}(\sin\theta - 1) + a.$$

To find minimum of $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$ for $\theta \in [0, \frac{\pi}{2}]$,

$$\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \frac{1-\tan\frac{\theta}{2}+1-1}{1+\tan\frac{\theta}{2}} = -1 + \frac{2}{1+\tan\frac{\theta}{2}}$$

whose minimum value is 0 corresponding to $\theta = \dfrac{\pi}{2}$.

Thus maximum value of $NM = a$ but in the sense that $CM,\ CN \to \infty$ such that $CN - CM \to a$.

Problem 10. To find the point on a given ellipse which is at the maximum distance from one end of the minor axis.

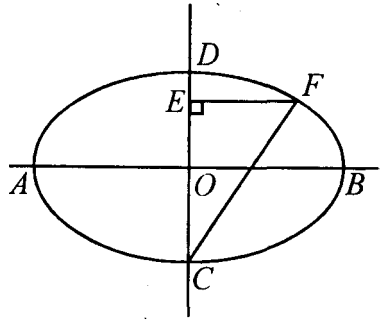


Fig. 1.10

Solution. Let the ellipse have the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, a > b$ referred to its major axis AB as x-axis and minor axis CD as y-axis (Fig.1.10).

Let F on the ellipse be at the maximum distance from the end C of its minor axis. Let the line through F parallel to AB meet CD at E. Let $CE = y$. $EF = x$-coordinate of F, $y = CE = b + OE = b + y$-coordinate of F.

Now, F being on the ellipse $\dfrac{FE^2}{a^2} + \dfrac{OE^2}{b^2} = 1$

or $$\frac{FE^2}{a^2} + \frac{(CE - CO)^2}{b^2} = 1\,,$$

i.e., $$\frac{FE^2}{a^2} + \frac{CE^2 - 2CO \cdot CE}{b^2} + 1 = 1$$

or $$\frac{FE^2}{a^2} + \frac{CE^2}{b^2} - \frac{2 \cdot CE}{b} = 0\,.$$

But, by Pythagoras theorem $CF^2 = CE^2 + EF^2$. Thus

$$\begin{aligned} CF^2 &= y^2 + \frac{2a^2y}{b} - \frac{a^2y^2}{b^2} \\ &= \left(1 - \frac{a^2}{b^2}\right) y^2 + \frac{2a^2}{b} y \\ &= \frac{2a^2}{b} y - \frac{a^2 - b^2}{b^2} y^2\,. \end{aligned}$$

If r is the maximum value of CF, then for this value,

$$r^2 = \frac{2a^2}{b} y - \frac{a^2 - b^2}{b^2} y^2$$

or $$\begin{aligned} \frac{b^2}{a^2 - b^2} r^2 &= \frac{2a^2b}{a^2 - b^2} y - y^2 \\ &= \left(\frac{a^2b}{a^2 - b^2}\right)^2 - \left(y - \frac{a^2b}{a^2 - b^2}\right)^2\,. \end{aligned}$$

Thus, the minimum value of a square being zero, the maximum value of $\dfrac{b^2}{a^2 - b^2} r^2$ is $\left(\dfrac{a^2b}{a^2 - b^2}\right)^2$ or, equivalently, the maximum value of r is $\dfrac{a^2b}{a^2 - b^2} \cdot \dfrac{\sqrt{a^2 - b^2}}{b} = \dfrac{a^2}{\sqrt{a^2 - b^2}}$.

Note: 1. This maximum is reached when $y = \dfrac{a^2b}{a^2 - b^2}$ which is

clearly $\leq 2b$. i.e., $\frac{a^2b}{a^2-b^2} \leq 2b$, i.e., $a^2 \leq 2(a^2-b^2)$ or $a^2 \geq 2b^2$. Thus the solution does not apply when $b < a < \sqrt{2}\,b$.

2. If $a \leq \sqrt{2}b$, $\frac{a^2b}{a^2-b^2} \geq 2b$ while y has range from 0 to $2b$; in this case the maximum value of r corresponds to $y = 2b$ i.e., $r = 2b$.

Problem 11. To find the triangle ABC with maximum angle A when the length of BC and of the altitude AD, from A to BC are fixed.

Solution. (see Fig. 1.11).
Let $BC = c$, $AD = p$ (fixed).
Let $B\hat{A}D = \alpha_1$, $D\hat{A}C = \alpha_2$.
Let $BD = x$ so that $DC = c - x$. Now,

$$\tan\alpha_1 = \frac{x}{p},\ \tan\alpha_2 = \frac{c-x}{p}$$

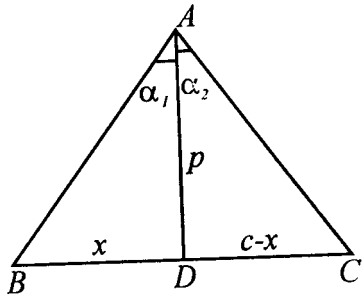


Fig. 1.11

and

$$\tan A = \tan(\alpha_1+\alpha_2) = \frac{\tan\alpha_1+\tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}$$

$$= \frac{\frac{x}{p}+\frac{c-x}{p}}{1-\frac{x(c-x)}{p^2}}$$

$$= \frac{cp}{p^2-cx+x^2}$$

If $0 < A < \frac{\pi}{2}$, A is maximum when $\tan A$ is maximum, since the tangent function is strictly increasing in this range. $\tan A$ is maximum when $r = \frac{p^2-cx+x^2}{cp}$ is minimum.

If r is this minimum value, which is positive supposedly, $x^2 - cx = rpc - p^2$, i.e., $x^2 - cx + \frac{c^2}{4} - \frac{c^2}{4} + p^2 = rpc$. i.e., $\frac{1}{pc}\left(x - \frac{c}{2}\right)^2 + \frac{1}{pc}\left(p^2 - \frac{c^2}{4}\right) = r$. A square being always non-negative the minimum value of r is $\frac{1}{4pc}(4p^2 - c^2)$. Hence the maximum value of $\tan A$ is $\frac{4pc}{4p^2 - c^2}$ if $c^2 \neq 4p^2$ or $c \neq 2p$ in which case $A = \frac{\pi}{2}$ (as is seen geometrically). The maximum is reached, if this case is excluded, when $x = \frac{c}{2}$, i.e., when the triangle is isosceles.

Depending on either $c < 2p$ or $c > 2p$, A is acute or obtuse respectively. When $c = 2p$, the minimum value of $\frac{p^2 - cx + x^2}{cp}$ which is $\cot A$ is 0 so that $A = \frac{\pi}{2}$.

Problem 12. Given a line segment AB and a straight line l not containing it to find the point D on l at which AB subtends the maximum angle.

Solution. Let AB or AB produced meet l at C (Fig. 1.12) assuming that AB and l are not parallel. If D is the position of the desired point on l, let E be the foot of the perpendicular from D on AB.

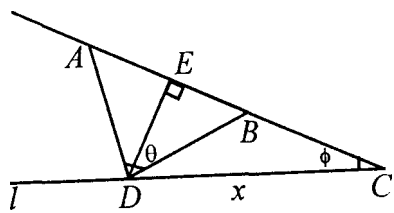


Fig. 1.12

Let $AC = a$, $BC = b$ constants. Let $CD = x$. Now,

$$\tan A\hat{D}B = \tan(A\hat{D}E + B\hat{D}E) = \frac{\tan A\hat{D}E + \tan B\hat{D}E}{1 - \tan A\hat{D}E \tan B\hat{D}E}$$

$$= \frac{\frac{AE}{DE} + \frac{BE}{DE}}{1 - \frac{AE}{DE} \cdot \frac{BE}{DE}} = \frac{(AE + BE)DE}{(DE)^2 - AE \cdot BE}$$

$$= \frac{AB \cdot DE}{(DE)^2 - AE \cdot BE}$$

Note that $DE = x \sin\phi$, if $B\hat{C}D = \phi$ (const).

$AE = a - x\cos\phi$, $BE = x\cos\phi - b$. Hence

$$\tan A\hat{D}B = \frac{(a-b)x\sin\phi}{x^2\sin^2\phi - (a - x\cos\phi)(x\cos\phi - b)}$$

$$= \frac{(a-b)x\sin\phi}{x^2 - (a+b)x\cos\phi + ab}$$

As the tangent is an increasing function, maximum of $A\hat{D}B$ is reached when $\tan D$ is maximum (provided the maximum exists). a, b, ϕ being constants the maximum is reached when $(x^2 - x\cos\phi(b+a) + ab)/x$ is minimum. If this minimum value is r

$$x^2 - x(a+b)\cos\phi + ab = rx$$

or

$$x^2 - \{(a+b)\cos\phi - r\}x + ab = 0$$

i.e.,

$$[x - \{(a+b)\cos\phi - r\}/2]^2 - \frac{\{(a+b)\cos\phi - r\}^2}{4} + ab = 0$$

i.e.,

$$\{(a+b)\cos\phi - r\}^2 = [x - \{(a+b)\cos\phi - r\}/2]^2 + ab$$

When the first term on the right hand is nonzero the l.h.s. is greater than ab. So, the minimum value of the r.h.s. is ab when $x - \{(a+b)\cos\phi - r\}/2 = 0$ and then r is also minimum with $\{(a+b)\cos\phi - r\}/4]^2 = ab$. i.e., $x = \{(a+b)\cos\phi - r\}/2 = \sqrt{ab}$. Maximum of $\tan D = \dfrac{\sin\phi}{(a+b)\cos\phi - 2\sqrt{ab}}$.

Remarks. 1. If suffices to consider ϕ to be an acute angle since if ϕ is obtuse the position for D when $A\hat{D}B$ is maximum clearly occurs when D lies on the arm of l containing the supplementary angle (see Fig. 1.12(a)).

2. When AB and l are parallel, (see Fig. 1.12(b)) the problem reduces to Problem 11.

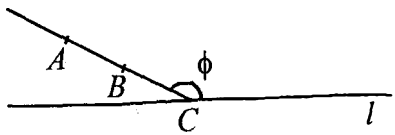


Fig. 1.12(a)

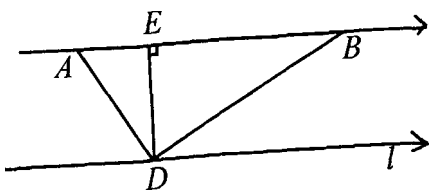


Fig. 1.12(b)

3. Note that the solution to Problem 12 identifies D geometrically as the point of contact of l with the circle passing through A, B and touching l.

Problem 13. To find the shortest line segment with its ends on two sides of a given triangle which bisects the triangle.

Solution. Let in ΔABC (see Fig. 1.13), PQ with P on BC and Q and CA be the shortest line such that ΔCPQ and quadrilateral $PQBA$ have equal area.

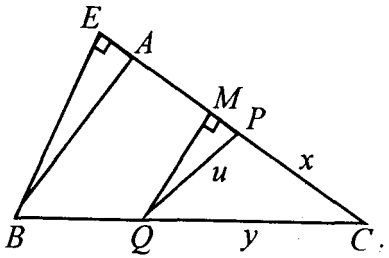


Fig. 1.13

Let $CP = x$, $CQ = y$, $PQ = u$, all varying with P, Q. Draw QM, BE perpendicular to CA meeting it at M, E respectively. $\Delta CQM|||\Delta CBE$ so that

$$\frac{QM}{CQ} = \frac{BE}{BC} = \sin C.$$

So, $QM = y \sin C$. Also $BE = a \sin C$ (a, b, c are lengths of the sides of ΔABC as usual). Hence

$$2\frac{CP \times QM}{2} = \frac{2xy \sin C}{2} = \frac{CA \times BE}{2} = \frac{ab \sin C}{2}$$

if PQ divides ΔABC into two halves, $xy = \frac{ab}{2}$, or equivalently $2xy = ab$ or $y = \frac{ab}{2x}$. By the cosine formula applied to ΔCPQ

$$\begin{aligned} u^2 = PQ^2 &= CP^2 + CQ^2 - 2CP \cdot CQ \cos C \\ &= x^2 + y^2 - 2xy \cos C = x^2 + \frac{a^2b^2}{4x^2} - ab \cos C \end{aligned}$$

If this is equal to r

$$rx^2 = x^4 - (ab \cos C)x^2 + \frac{a^2b^2}{4}$$

or $$x^4 - (ab\cos C + r)x^2 + \frac{a^2b^2}{4} = 0$$

i.e., $$\left\{x^2 - \frac{ab\cos C + r}{2}\right\}^2 + \frac{a^2b^2}{4} - \frac{(ab\cos C + r)^2}{4} = 0$$

i.e., $$\left\{x^2 - \frac{ab\cos C + r}{2}\right\}^2 + \frac{a^2b^2}{4} = \frac{(ab\cos C + r)^2}{4}.$$

Hence, the minimum value of the r.h.s. is reached when the first term on the l.h.s. is 0. i.e., $x^2 = \frac{ab\cos C + r}{2}$ and then $r = ab(1 - \cos C)$. So $x^2 = \frac{ab}{2}$ or $x = \sqrt{\frac{ab}{2}}$; also $y = \frac{ab}{2\sqrt{\frac{ab}{2}}} = \sqrt{\frac{ab}{2}}$.

Length of $$PQ^2 = \frac{ab}{2} + \frac{ab}{2} - ab\cos C = ab(1 - \cos C)$$

$$= ab\left\{\frac{2ab + c^2 - a^2 - b^2}{2ab}\right\} = \frac{c^2 - (a-b)^2}{2}$$

so that $PQ = \sqrt{\frac{(c - a + b)(c + a - b)}{2}}$. To find the least such PQ, find $a + b - c$, $b + c - a$, $c + a - b$ and choose the least and next higher of them. e.g., if $a + b - c$, $b + c - a$ are these two in that order, P, Q are to be chosen on BC, BA such that BP, $BQ = \sqrt{\frac{ca}{2}}$ and the length of PQ is $\sqrt{\frac{(a + b - c)(b + c - a)}{2}}$.

Problem 14. To find maximum of $y = x\tan\theta - \frac{x^2}{4p\cos^2\theta}$ when x varies, p, θ being constants.

Solution. Given $y = \frac{4xp\cos^2\theta\tan\theta - x^2}{4p\cos^2\theta}$, it suffices to find the maximum of $4xp\cos^2\theta\tan\theta - x^2$, p, θ being constants. Writing $2p\cos^2\theta\tan\theta = A$, we are to maximize $2Ax - x^2 = r$. Now $x^2 - 2Ax + A^2 = A^2 - r$. i.e., $(x - A)^2 = A^2 - r$ or

$r = A^2 - (x - A)^2$ which is maximum when $x - A = 0$, i.e., $x = A = 2p\cos^2\theta\tan\theta$.

Then $y = 2p\cos^2\theta\tan\theta \cdot \tan\theta - \frac{4p^2\cos^4\theta\tan^2\theta}{4p\cos^2\theta} = 2p\sin^2\theta - p\sin^2\theta = p\sin^2\theta$.

Remark. Problem 14 arises in the study of the motion of a projectile.

Problem 15. Given a positive real number a, to write a as a product of two factors the sum of whose squares is maximum.

Solution. Let $a = x \cdot \frac{a}{x}$. To find x such that $x^2 + \frac{a^2}{x^2}$ is maximum. If $x^2 + \frac{a^2}{x^2} = r$, then $x^4 - rx^2 = a^2, x$ being nonzero. i.e., $a^2 - \left(x^2 - \frac{r}{2}\right)^2 = \frac{r^2}{4}$ so that r is maximum when $x^2 = \frac{r}{2}$ and the maximum value of $\frac{r^2}{4}$ is a^2, i.e., $r = 2a$, $x^2 = a$ or $x = \sqrt{a}$.

Problem 16. To find $x, 0 < x < 1$, such that, $x - x^2$ is maximum.

Solution. If $x - x^2 = r$, i.e., $r = \frac{1}{4} - (x - \frac{1}{2})^2$ so that r is maximum when $x = \frac{1}{2}$.

Problem 17. To find the triangle of maximum area with given length of one side and its perimeter.

Solution. Let in ΔABC, with the usual notation, let a be given and $s = \frac{a+b+c}{2}$. Then the square of area of $\Delta ABC = s(s-a)(s-b)(s-c)$. Now, $b + c = 2s - a$ so that $c = 2s - a - b$. Denoting $(s-b)(s-c) = r', a, s$ being

constants, we are to maximize r'. $r' = s^2 - (b + c)s + bc = s^2 - (2s - a)s + b(2s - a - b)$. For the same reason, it is enough to maximize $b(2s - a - b)$ when b varies. If $r = b(2s - a - b) = b(2s - a) - b^2 = \left(\frac{2s - a}{2}\right)^2 - \left(b - \frac{2s - a}{2}\right)^2$. The maximum value of r is $\left(\frac{2s - a}{2}\right)^2$ when $b = \frac{2s - a}{2}$. Then $c = 2s - a - \left(\frac{2s - a}{2}\right) = \frac{2s - a}{2}$. i.e., ΔABC needs to be isosceles.

Problem 18. To inscribe in ΔABC a parallelogram one of whose vertices is A such that it has maximum area.

Solution.

Let in ΔABC, $ADEF$ be an inscribed parallelogram as shown in Fig. 1.14.

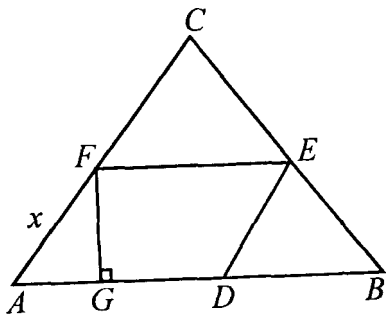


Fig. 1.14

Let FG be perpendicular to AB. Let $AF = x$.

$\Delta BED|||\Delta BCA$ so that

$$\frac{DE}{AC} = \frac{BE}{BC} = \frac{BD}{AB};$$

$$\frac{AB - AD}{c} = \frac{DE}{b} = \frac{AF}{b} = \frac{x}{b}$$

i.e., $\frac{c - AD}{c} = \frac{x}{b}$, i.e., $b(c - AD) = cx$. $AD = \frac{c(b - x)}{b}$.

Area of parallelogram

$$ADEF = AD \times FG = \frac{c(b - x)}{b} \times x \sin A = \frac{c}{b}$$
$$(bx \sin A - x^2 \sin A).$$

$b, c, \sin A$ being constants it is enough to maximize $bx - x^2$.

$bx - x^2 = \frac{b^2}{4} - (x - \frac{b}{2})^2$, the maximum value being $\frac{b^2}{4}$ when $x = \frac{b}{2}$. $AD = \frac{c(b - \frac{b}{2})}{b} = \frac{c}{2}$: D is the midpoint of AB.

Problem 19. To find the parallelogram of maximum area one of whose angles and the perimeter are fixed.

Solution. If $ABCD$ is the parallelogram, let $\hat{A}$ be be fixed and let E be the foot of the perpendicular from D to AB (see Fig. 1.15). If s is the semiperimeter $AB + AD = s$, constant. Let $AD = x$.

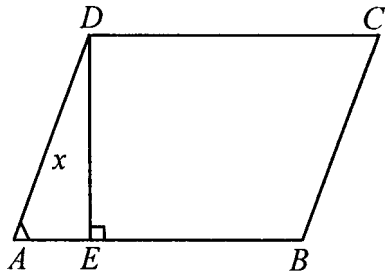


Fig. 1.15

$$\begin{aligned} \text{Area of } ABCD &= AB \times DE. \\ &= (s - AD) \times AD \times \sin A \\ &= (s - x) \times x \sin A \\ &= \sin A(sx - x^2) \end{aligned}$$

As in Problem 18 $sx - x^2$ is maximum when $x = \frac{s}{2}$ and the maximum value is $\frac{s^2}{4}$. Thus the parallelogram has maximum area when its sides are equal. Its area is then $\frac{s^2}{4} \sin A$.

Problem 20. To find the triangle of maximum perimeter with one of its sides and the angle opposite to it fixed.

Solution. (See Fig. 1.16). Let in ΔABC, $BC = a$ and angle A be fixed. Let $AB = x$, $AC = y$. Then to find the maximum of $a + x + y = 2s$ or equivalently of $a+x+y$ or of $x+y$.

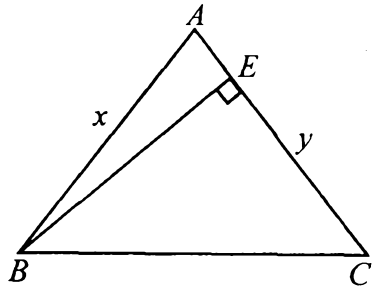


Fig. 1.16

Now, by the cosine formula $a^2 = x^2 + y^2 - 2xy\sin A$. Let BE be the altitude from B. Then $AE = x\cos A$, $BE = x\sin A$, $EC^2 = a^2 - BE^2 = a^2 - x^2\sin^2 A$. Thus $y = AC = x\cos A + \sqrt{a^2 - x^2\sin^2 A}$. Hence $x + y = x(1+\cos A)+\sqrt{a^2 - x^2\sin^2 A} = r$, say. Then $r-x(1+\cos A) = \sqrt{a^2 - x^2\sin^2 A}$. Squaring both sides $r^2 - 2xr(1 + \cos A) + x^2(1 + \cos A)^2 = a^2 - x^2\sin^2 A$

or $$r^2 - 2xr(1 + \cos A) + 2x^2 + 2x^2\cos A = a^2$$

i.e., $$r^2 + 2(1 + \cos A)(\tfrac{r}{2} - x)^2 - 2(1 + \cos A)\tfrac{r^2}{4} = a^2$$

i.e., $$r^2(\tfrac{1}{2} - \tfrac{\cos A}{2}) = a^2 - 2(1 + \cos A)(\tfrac{r}{2} - x)^2.$$

Hence r is maximum when $x = \frac{r}{2}$, i.e., $x = y$. Also $x + y = r = \sqrt{2}a/\sqrt{1 - \cos A}$. In other words, the maximum perimeter is obtained when the triangle is isosceles.

Problem 21. To insert a rectangle of maximum area in a semicircle with one of the sides of the rectangle lying along the bounding diameter of the semicircle.

Solution. Let $PQRS$ be a rectangle inscribed in the given

semicircle with centre C and radius a (See Fig. 1.17a).

Draw the perpendicular to the bounding diameter at C meeting SR at E. Then CE, being a diameter perpendicular to the chord SR, bisects it. If $CE = x$, then $PQ = 2PC = 2\sqrt{a^2 - x^2}$. Area of $PQRS$ $= x \times 2\sqrt{a^2 - x^2} = 2x\sqrt{a^2 - x^2}$. If $r = x\sqrt{a^2 - x^2}$, then $r^2 = x^2(a^2 - x^2) = a^2x^2 - x^4$. We are to maximize $r^2 = \frac{a^4}{4} - (x^2 - \frac{a^2}{2})^2$ which is maximum when $x^2 - \frac{a^2}{2} = 0$ or $x = \frac{a}{\sqrt{2}}$. Then $PQ = 2\sqrt{a^2 - \frac{a^2}{2}} = 2\sqrt{\frac{a^2}{2}} = \sqrt{2}a$. $PQRS$ is constructed by choosing P, Q on the diameter on either side of C at distance of $\frac{a}{\sqrt{2}}$ from C. Then erect the perpendiculars at P, Q to meet the semicircle at S, R respectively.

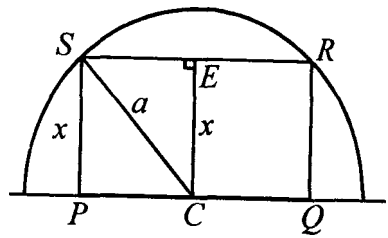


Fig. 1.17(a)

Note: If three of the vertices, PQR of rectangle $PQRS$ lie on the semicircular arc (see Fig. 1.17b), then PR is a diameter, which is impossible.

If all the four lie on the semicircle, then the rectangle will be degenerate so that one need not explicitly state that two of the vertices lie on the diameter.

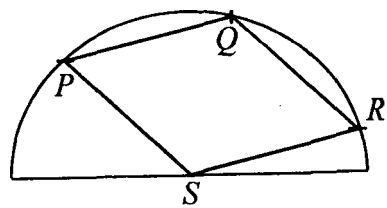


Fig. 1.17(b)

Problem 22. To find the square of least area inscribed in a given square, i.e., with vertices one on each of the four sides of the given square.

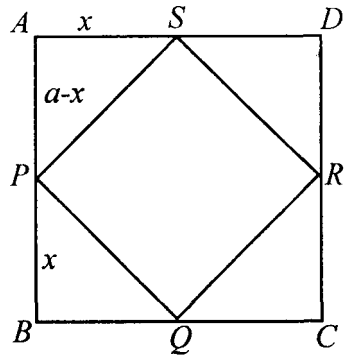


Fig. 1.18

Solution. Let $ABCD$ be the given square and $PQRS$ the square desired (Fig. 1.18). Let the side of square $ABCD$ be of length a. Let $PB = x$. $PQRS$ being a square $PQ = PS$. Also, $A\hat{P}S + B\hat{P}Q = \frac{\pi}{2} = A\hat{S}P + A\hat{P}S$. So, $B\hat{P}Q = A\hat{S}P$. Hence the right angled triangles APS and BQP are congruent so that $AS = BP = x$. From right angled triangle APS, by Pythagoras theorem $PS^2 = x^2 + (a-x)^2 = a^2 - 2ax + 2x^2 = a^2 - \frac{a^2}{2} + 2(x - \frac{a}{2})^2 = \frac{a^2}{2} + 2(x - \frac{a}{2})^2$. Thus PS is the least when $x = \frac{a}{2}$ and $PS = \frac{a}{\sqrt{2}}$. ΔAPS is congruent to ΔDSR so that $PS = SR$. i.e., P, Q, R, S are midpoints of the sides AB, BC, CD, DA of the given square and the square of least area $PQRS$ has side of length $\frac{a}{\sqrt{2}}$.

Remark. What has been actually shown is that the rectangle of least area inscribed in a square is a square whose side is $\frac{1}{\sqrt{2}}$ times the side of the given square.

Note that by choosing $AP = BQ = CR = DS = x$ for some $x, 0 < x < a$, we get a square $PQRS$ inscribed in $ABCD$.

Problem 23. To find the rectangle of largest area inscribed in a given ellipse with sides parallel to the axes of the ellipse.

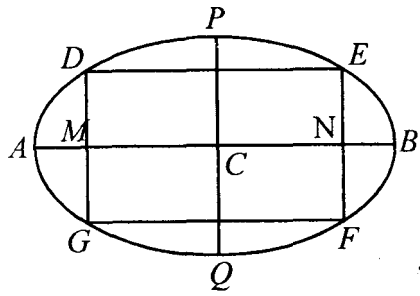


Fig. 1.19

Solution. Let AB, PQ be the major and minor axes of the given ellipse and C its centre (Fig. 1.19).

Let $DEFG$ be an inscribed rectangle. Let DG, EF meet AB at M and N respectively. Let $AC = a$, $PC = b$ (constants).

Let $CM = x$. Then $CN = x$ since $DG = EF$ and both are parallel to the minor axis PQ. Hence $MN = 2x$. D being a point on the ellipse

$$\frac{MC^2}{a^2} + \frac{MD^2}{b^2} = 1 \quad \text{or} \quad MD^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\text{i.e.,} \quad MD = \frac{b}{a}\sqrt{a^2 - x^2}.$$

$$\text{Area of rectangle } DEFG = 2CM \times 2MD$$
$$= 2x \times 2\frac{b}{a}\sqrt{a^2 - x^2}$$

Thus we are to maximize $x\sqrt{a^2 - x^2}$ or to maximize $x^2(a^2 - x^2) = a^2x^2 - x^4 = \frac{a^4}{4} - \left(x^2 - \frac{a^2}{2}\right)^2$. The maximum is reached when $x^2 = \frac{a^2}{2}$ or $x = \frac{a}{\sqrt{2}}$ and the maximum area $= 4\frac{a}{\sqrt{2}} \times \frac{b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{4}{\sqrt{2}}b\frac{a}{\sqrt{2}} = 2ab$ (cf. Solution of Problem 21).

Problem 24. Show that a triangle has maximum area given the length of a side and the angle opposite to it when it is isosceles.

Solution.

Let in ΔABC, $BC = a$ and $B\hat{A}C = \theta$ be given (see Fig. 1.20). Let BE be the altitude through B. Let $AB = x$, $AC = y$. Then $AE = x\cos\theta$, $BE = x\sin\theta$. $CE = \sqrt{a^2 - x^2\sin^2\theta}$. $y = AE + CE = x\cos\theta + \sqrt{a^2 - x^2\sin^2\theta}$.

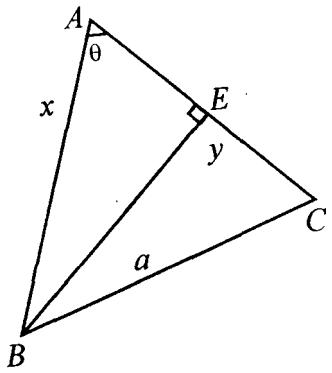


Fig. 1.20

$$\text{Area of } \Delta ABC = \Delta = \frac{1}{2}AC \cdot BE$$

$$= \frac{1}{2}x\sin\theta\left(x\cos\theta + \sqrt{a^2 - x^2\sin^2\theta}\right)$$

$$= \frac{1}{2}\sin\theta\left(x^2\cos\theta + x\sqrt{a^2 - x^2\sin^2\theta}\right)$$

We are to maximize Δ, θ being fixed, or to maximize $x^2\cos\theta + x\sqrt{a^2 - x^2\sin^2\theta} = r$, say. Then

$$(r - x^2\cos\theta)^2 = x^2(a^2 - x^2\sin^2\theta)$$

i.e., $\quad x^4 - x^2(2r\cos\theta + a^2) + r^2 = 0$.

i.e., $\quad \left(x^2 - \dfrac{2r\cos\theta + a^2}{2}\right)^2 - \left(\dfrac{2r\cos\theta + a^2}{2}\right)^2 + r^2 = 0$

i.e., $\quad r^2 = \left(\dfrac{2r\cos\theta + a^2}{2}\right)^2 - \left(x^2 - \dfrac{2r\cos\theta + a^2}{2}\right)^2$

so that r is maximum when the second term on the r.h.s. $= 0$

and so $r^2 = \left(\dfrac{2r\cos\theta + a^2}{2}\right)^2$, i.e., $r = \dfrac{2r\cos\theta + a^2}{2}$

i.e., $2r(1-\cos\theta) = a^2$, i.e., $4r\sin^2\dfrac{\theta}{2} = a^2$. Then

$$x^2 = \frac{2r\cos\theta + a^2}{2} = \frac{\frac{a^2}{2\sin^2\frac{\theta}{2}}\cos\theta + a^2}{2} = \frac{a^2\left(\cos\theta + 2\sin^2\frac{\theta}{2}\right)}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{a^2}{4\sin^2\frac{\theta}{2}}$$

so that $x = \dfrac{a}{2\sin\frac{\theta}{2}}$. This precisely means that the median from A is the altitude from A. So ABC is an isosceles triangle with $AB = AC$.

Problem 25. To construct a right triangle of least area such that its two sides are along the bounding radii of a quadrant of a circle and the third is a tangent to the circle.

Solution. (See Fig. 1.21). Let OAB be a quadrant of a circle of radius a and PQ the tangent at T on the circle meeting OA, OB produced at P, Q respectively.

Let $TM \perp OA, TN \perp OB$. The problem is to find T such that ΔPOQ has the least area among such triangles.

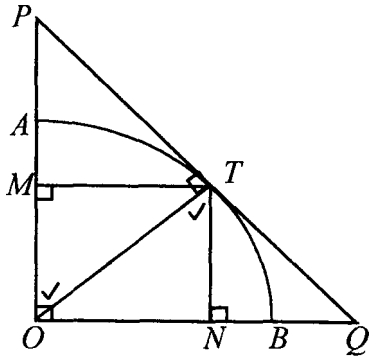


Fig. 1.21

It is a problem of minimizing $OP \times OQ$. $OT \perp PQ$ so that ΔOMT and ΔQTO are similar right triangles. Hence $\dfrac{OQ}{TO} =$

$\frac{OT}{TM}$ or $TM = \frac{OT^2}{OQ} = \frac{a^2}{x}$ if $OQ = x$.

Likewise similarity of Δs OTP and TNO gives $\frac{TN}{OT} = \frac{OT}{OP}$ or $TN = \frac{a^2}{y}$ if $OP = y$. Now, $OT^2 = TM^2 + TN^2$

i.e., $a^2 = \frac{a^4}{x^2} + \frac{a^4}{y^2}$ or $a^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 1$ or $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$,

i.e., $\frac{1}{y^2} = \frac{1}{a^2} - \frac{1}{x^2} = \frac{x^2 - a^2}{a^2x^2}$ or $y^2 = \frac{a^2x^2}{x^2 - a^2}$,

$y = \frac{ax}{\sqrt{x^2 - a^2}}$.

So $$OP \cdot OQ = \frac{ax^2}{\sqrt{x^2 - a^2}}.$$

It is enough to find minimum of $\frac{x^2}{\sqrt{x^2 - a^2}} = r$, say. Then $x^4 = r^2(x^2 - a^2)$, i.e., $x^4 - r^2x^2 = -a^2r^2$, i.e., $\left(x^2 - \frac{r^2}{2}\right)^2 = \frac{r^4}{4} - a^2r^2$ or $a^2r^2 + \left(x^2 - \frac{r^2}{2}\right)^2 = \frac{r^4}{4}$ so that r can be minimum if and only if $x^2 - \frac{r^2}{2} = 0$ i.e., $x = \frac{r}{\sqrt{2}}$.

i.e., $r = \sqrt{2}x$. Hence

$$\frac{x^2}{\sqrt{x^2 - a^2}} = \sqrt{2}x$$

i.e., $x^2 = 2(x^2 - a^2)$, i.e., $x^2 = 2a^2$, i.e., $x = \sqrt{2}a$.

$y = \frac{\sqrt{2}a^2}{\sqrt{2a^2 - a^2}} = \sqrt{2}a$. Thus $x = y = \sqrt{2}a$.

Remark. Note that PQ in the case of ΔOPQ having minimum area, is one of the sides of a circumscribing square of the circle.

Problem 26. A ship starts from a place A and moves in the direction of the ray AQ. A boat simultaneously starts from a place B with the objective of reaching the position of the ship or to be as near to the ship as possible. Given the speeds of both the ship and the boat, to determine the direction of the movement of the boat so that the objective is realized.

Solution. Let as in Fig.1.22, D, F be the positions of the ship and boat when they are at the minimum distance. Then DF should pass through B. For, if we take the circle centre B and radius BF on which alone F can lie at that particular time, for any other position F' of F other than on the ray BD (radius) DF' is strictly larger than DF.

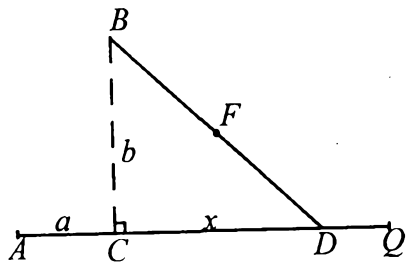


Fig. 1.22

Let $CD = x$, where C is the foot of the perpendicular from B on AQ. Let $AC = a$, $BC = b$ constants and let the speed of the ship bear to that of the boat the ratio $m : n$. Then $BD^2 = b^2 + x^2$ or $BD = \sqrt{b^2 + x^2}$. Also $\dfrac{AD}{BF} = \dfrac{m}{n}$. i.e., $BF = \dfrac{n}{m}AD = \dfrac{n}{m}(a + x)$. So $DF = DB - BF = \sqrt{b^2 + x^2} - \dfrac{n}{m}(a + x)$. To minimize DF, it suffices to get the minimum of $\sqrt{b^2 + x^2} - \dfrac{n}{m}x = r$ (n, m, a being constants), say.

Then $b^2 + x^2 = \left(r + \dfrac{n}{m}x\right)^2 = r^2 + \dfrac{2nr}{m}x + \dfrac{n^2}{m^2}x^2$.

i.e., $$x^2\left(-\frac{n^2}{m^2}+1\right)-\frac{2nr}{m}x=r^2-b^2$$

or $$x^2-\frac{2nrm}{(m^2-n^2)}x=m^2\frac{r^2-b^2}{m^2-n^2}$$

or $$\left(x-\frac{nrm}{m^2-n^2}\right)^2 = \frac{m^2(r^2-b^2)}{m^2-n^2}+\frac{n^2m^2r^2}{(m^2-n^2)^2}$$

$$= \frac{m^2(m^2-n^2)(r^2-b^2)+n^2m^2r^2}{(m^2-n^2)^2}$$

$$= \frac{m^4(r^2-b^2)+m^2n^2b^2}{(m^2-n^2)^2}$$

$$= \frac{m^4r^2-m^2b^2(m^2-n^2)}{(m^2-n^2)^2}$$

i.e., $$\frac{m^4r^2}{(m^2-n^2)^2}=\left(x-\frac{nrm}{m^2-n^2}\right)^2+\frac{m^2b^2}{m^2-n^2}$$

so that r can be minimum only when $x=\dfrac{nrm}{m^2-n^2}$,

i.e., $r=\dfrac{(m^2-n^2)x}{nm}$; i.e., when $\sqrt{b^2+x^2}-\dfrac{n}{m}x=\dfrac{m^2-n^2}{mn}x$.

(Note: Compare Problem 9.)

$$\sqrt{b^2+x^2} = \frac{m^2-n^2+n^2}{mn}x=\frac{m^2}{mn}x=\frac{m}{n}x$$

i.e., $$b^2+x^2 = \frac{m^2}{n^2}x^2 \quad \text{i.e.,} \quad \frac{m^2-n^2}{n^2}x^2=b$$

i.e., $$x = \frac{nb}{\sqrt{m^2-n^2}}$$

Remarks. It has been tacitly assumed in the solution that B does not lie on AQ and that $m \gtrless n$. If B lies on AQ to the right of A and $AB=a$, $AD=x$, then $BF=\frac{n}{m}x$ so that $DF=AD-AB-BF=x-a-\frac{n}{m}x=(1-\frac{n}{m})x-a$

so that when $n < m$, DF increases as x increases. If $n > m$, then F lies to the right of D and FD increases as x increases. In either case there is no maximum but the minimum is reached in these cases when $x = 0$, i.e., the initial position. If $n = m, DF$ is constantly equal to AB. If B lies to the left of A on AQ, the situation can be discussed similarly. When $n = m$, $DF = \sqrt{b^2 + x^2} - (a + x)$ so that it suffices to minimize $\sqrt{b^2 + x^2} - x = r$. Then $(x+r)^2 = b^2 + x^2$, so that the minimum value of $r = b$ is reached when $x = 0$. Then $DF = b - a$. The direction in which the boat moves is perpendicular to the direction of the ship in this case.

Problem 27. If a, b are constants, to find $\max\{b - (x-a)^2\}$.

Solution. If $b - (x - a)^2 = r$, r has a maximum if and only if $x - a = 0$ or $x = a$.

Problem 28. To find positive x for which $\dfrac{x}{1 + x^2}$ is maximum.

Solution. Equivalently to find x for which $\dfrac{1 + x^2}{x}$ is minimum. If $\dfrac{1 + x^2}{x} = r$, then $x^2 - xr = -1$, i.e., $\left(x - \dfrac{r}{2}\right)^2 = \dfrac{r^2}{4} - 1$ or $\dfrac{r^2}{4} = \left(x - \dfrac{r}{2}\right)^2 + 1$. So, r is minimum when $x = \dfrac{r}{2}$ or $r = 2x$. i.e., $x^2 - 2x^2 = -1$, i.e., $x = 1$.

Problem 29. a, b, c being given non-negative real numbers, to find x for which $a^4 + b^3x - c^2x^2$ is maximum; given $b > 0$.

Solution. If $c = 0$, there is no maximum. When $c \neq 0$, $a^4 + b^3x - c^2x^2 = c^2\left(\dfrac{a^4}{c^2} + \dfrac{b^3}{c^2}x - x^2\right)$. It suffices to maximize, since a, b, c are constants, $\dfrac{b^3}{c^2}x - x^2 = r$, say.

$r = x\left(\frac{b^3}{c^2} - x\right)$. The question is reduced to maximizing a product of two quantities whose sum is fixed, viz., $\frac{b^3}{c^2}$. The solution is (see Problem 1) $x = \frac{b^3}{2c^2}$.

Problem 30. To find a value of x for which $a + \sqrt[3]{a^3 - 2a^2x + ax^2}$, $a \geq 0$, is minimum.

Solution. It is enough to find maximum of $\sqrt[3]{a^3 - 2a^2x + ax^2}$ or equivalently of $a^3 - 2a^2x + ax^2$ or for what matters, of $x^2 - 2ax$ since a is constant; i.e., of $x(x-2a)$. If $x(x-2a) = x^2 - 2ax = r$ or $x(2a-x) = -r$. The product on the l.h.s. is maximum when $x = \frac{2a}{2} = a$. Hence r is minimum when $x = a$. The minimum value is a.

Problem 31. Given a fixed number $a \geq 0$ to find x such that $\frac{a^2x}{(a-x)^2}$ is maximum.

Solution. $\frac{a^2x}{(a-x)^2}$ is maximum when $\frac{(a-x)^2}{a^2x}$ is minimum, i.e., when $\frac{(a-x)^2}{x}$ is minimum. If $\frac{(a-x)^2}{x} = r$, then $(a-x)^2 = rx$; i.e., $a^2 - 2ax + x^2 = rx$. i.e., $a^2 - (2a+r)x + x^2 = 0$ or $a^2 + \left(x - \frac{2a+r}{2}\right)^2 = \left(\frac{2a+r}{2}\right)^2 \cdot \left(\frac{2a+r}{2}\right)^2$ and so $\frac{2a+r}{2}$ and so r is minimum when $x = \frac{2a+r}{2}$ or $r = 2(x-a)$. Then $\frac{(a-x)^2}{x} = 2(x-a)$. If $x \geq a$, $x = a$ or $x - a = +2x$ or $x = a$. If $x \leq a$, $\frac{(a-x)^2}{x} = -2(a-x)$, i.e., $x = a$ or $a - x = -2x$, $x = a$ or $x = \frac{a}{3}$. When $x = a$ the

minimum value of r is 0. If $x = \frac{a}{3}$ the value of r is greater than 0 unless $a = 0$ (in which case the function itself is 0). Thus $x = a$ gives the maximum.

Problem 32. To determine the pair of conjugate diameters of an ellipse which include the largest angle.

Solution. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let the conjugate diameters to be determined be of length $2x$ and $2x'$. It is known (see e.g. S.L.Loney, Coordinate Geometry (Part-I), p.257) that $x^2 + x'^2 = a^2 + b^2$ and that if θ is the angle included by the conjugate diameters, then $xx' \sin\theta = ab$ or $\sin\theta = \frac{ab}{xx'} = \frac{ab}{x\sqrt{a^2+b^2-x^2}}$. The problem reduces to finding the minimum value of $\sin\theta$ since $\sin\theta$ decrease from 1 to 0 in $\left[\frac{\pi}{2}, \pi\right]$. This, again, reduces to finding x corresponding to maximum of $x\sqrt{a^2+b^2-x^2}$. If $r = x\sqrt{a^2+b^2-x^2}$, then $r^2 = x^2(a^2+b^2-x^2)$ is maximum under the condition $x^2 + (a^2 + b^2 - x^2) = a^2 + b^2$, constant, when $x^2 = a^2 + b^2 - x^2$ or $2x^2 = a^2 + b^2$ or $x = \sqrt{\frac{a^2+b^2}{2}}$. $x' = \sqrt{(a^2+b^2) - \frac{a^2+b^2}{2}} = \sqrt{\frac{a^2+b^2}{2}}$ and $\sin\theta = \frac{2ab}{(a^2+b^2)}$.

Problem 33. If $x, y \geq 0$, $y^2 - 2mxy + x^2 = a^2$, where $0 < m < 1$, a are constants, to find a value of x for which y takes its maximum value.

Solution. $x^2 - 2mxy = a^2 - y^2$, i.e., $(x - my)^2 = a^2 - (1 + m^2)y^2$ or $(1 + m^2)y^2 = a^2 - (x - my)^2$

which shows that y is maximum when $x = my$ or $y = \frac{x}{m}$. Then $\frac{x^2}{m^2} - 2mx \cdot \frac{x}{m} + x^2 = a^2$; i.e., $\left(-1 + \frac{1}{m^2}\right) x^2 = a^2$ or $\frac{(1-m^2)}{m^2} x^2 = a^2$ or $x = \frac{am}{\sqrt{1-m^2}}$.

Remarks. When $m = 0$, $y = \sqrt{a^2 - x^2}$ which is maximum when $x = 0$. When $m = 1$, $|y - x| = a$ so that there is no maximum for y. When $a = 0$, the maximum value of y is 0 reached when $x = my$ so that $x = 0$.

Problem 34. To inscribe a rectangle of largest area in a circle.

Solution. Let $ABCD$ be desired rectangle inscribed in a circle centre O. Let the radius of the circle be a. Let EF be the diameter bisecting the side AB of the rectangle at G.

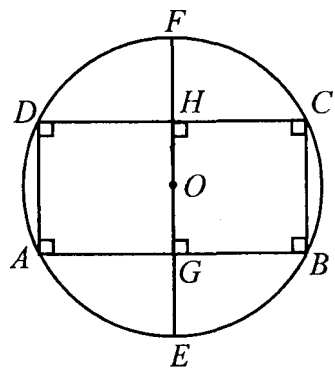


Fig. 1.23

Let $OG = x$ (see Fig. 1.23). Then EF is perpendicular to both AB and CD (parallel chords) and bisects CD too, say at H. Hence $EF||BC$. $GBCH$ is a rectangle and so $GH = BC$. But $GH = 2OG$ since the equal chords AB, CD are at equal distances $OG = OH$ from O. If $OG = x$, then $BC = 2OG = 2x$ and $GB = OB^2 - OG^2 = a^2 - x^2$, $AB = 2\sqrt{a^2 - x^2}$. We have, therefore, area of rectangle $ABCD = AB \times BC = 2\sqrt{a^2 - x^2} \cdot 2x$. Thus, it is enough

to maximize $x\sqrt{a^2 - x^2}$. As in solution of Problem 21, the maximum is reached when $x = \frac{a}{\sqrt{2}}$. $BC = 2x = 2\frac{a}{\sqrt{2}} = \sqrt{2}a$. $AB = 2\sqrt{a^2 - \frac{a^2}{2}} = 2\frac{a}{\sqrt{2}} = \sqrt{2}a$. i.e., the area is maximum when the rectangle is a square.

Problem 35. Given an angle and a point not on its arms to find a straight line, passing through that point, which makes a triangle of minimum area with the angle.

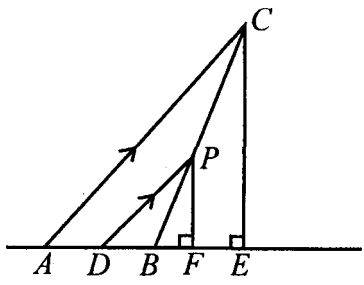


Fig. 1.24

Solution. Let $C\hat{A}B$ be the given angle and let P lie in the angle (Fig. 1.24).

Let CPB be the desired line. Let E be the foot of the perpendicular from C on AB, F the foot of the perpendicular from P on AB. Let $PD||CA$. Then PD, PF, AD are constants, given. Let $AD = a$, $DP = b, PF = c$ and let $AB = x$ (variable).

Δs BPD, BCA are similar. So, $\frac{BD}{DP} = \frac{BA}{AC}$,

i.e., $AC = \frac{BA \cdot DP}{BD}$. Δs DPF and ACE are similar.

So, $\frac{DP}{PF} = \frac{AC}{CE}$, i.e., $AC = \frac{DP \times CE}{PF}$.

Hence $\frac{BA}{BD} = \frac{AC}{DP} = \frac{DP \times CE}{PF \times DP} = \frac{CE}{PF}$ or $BD = \frac{BA \times PF}{CE}$

or $\frac{BD}{PF} = \frac{BA}{CE}$ i.e., $\frac{x-a}{c} = \frac{x}{CE}$ or $CE = \frac{cx}{(x-a)}$.

Area of $\Delta ABC = \frac{1}{2} AB \times CE = \frac{1}{2} x \cdot \frac{cx}{x-a} = \frac{1}{2} \frac{cx^2}{x-a}$. It is sufficient to minimize $\frac{x^2}{x-a}$. If $\frac{x^2}{x-a} = r$, then $x^2 = rx - ar$, i.e., $\left(x - \frac{r}{2}\right)^2 = \frac{r^2}{4} - ar$ or $\frac{r^2}{4} = (x - \frac{r}{2})^2 + ar$. Hence r is minimum when $x = \frac{r}{2}$, i.e., $r = 2x$. Then $\frac{x^2}{x-a} = 2x$, i.e., $x^2 = 2x^2 - 2ax$, i.e., $x^2 - 2ax = 0$, i.e., $x(x - 2a) = 0$. i.e., $x = 0$ or $x = 2a$.

When $x = 0$, no triangle is formed since CP passes through A. When $x = 2a$, to construct the triangle, draw parallel to one of the arms to meet the other arm at D. If A is the vertex of the angle take twice the distance AD on this arm to get B. Join BP and extend it to meet the other arm at C. ABC is a triangle of minimum area. Its area is $\frac{1}{2} \frac{4a^2 c}{2a - a} = 2ac$.

Remarks. (i) If the angle is 90° (Fig. 1.24(a)), let $AB = x$. $PF = a$, $AF = b$ are constants. Δs BPF, BCA are similar triangles. So

$$\frac{FB}{PF} = \frac{AB}{AC}$$

If $AB = x$, then $\frac{x-b}{a} = \frac{x}{AC}$ or $AC = \frac{ax}{x-b}$. Area of $\Delta ABC = \frac{1}{2} \frac{ax^2}{x-b}$, so that it suffices to minimize $\frac{x^2}{x-b}$. This occurs, as in the solution of Problem 35, when $x = 2b$, i.e., B is chosen such that F is the midpoint of AB.

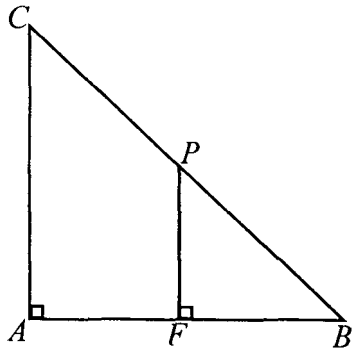


Fig. 1.24(a)

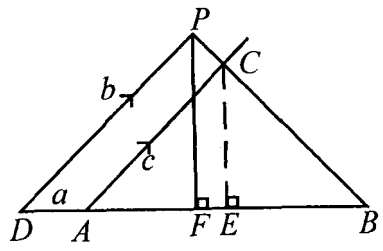


Fig. 1.24(b)

(ii) If P lies outside the angle (see Fig. 1.24(b)), the similarity of pairs of triangles mentioned in the solution is in order and we have $\frac{BD}{PF} = \frac{BA}{CE}$ but in this case $BD = AD + AB = a + x$ so that it suffices to minimize $\dfrac{x^2}{x+a} = r$ say. Then $\left(x - \dfrac{r}{2}\right)^2 = \dfrac{r^2}{4} + ar$ so that minimum of r occurs when $x = 0$ or $x = \dfrac{r}{2}$ in this case too. When $x = \dfrac{r}{2}$, $x = -2a$ so that B should be chosen to the left of D with $DB = a$. This corresponds to the angle supplementary to the given angle. If we were to choose the angle as such only $x = 0$ applies and the minimum is the triangle at the point A of zero area.

Problem 36. ΔABC is right angled at B. Given a line l in the plane of the triangle rotated around B in the plane, to find its position so that the sum of the heights of A and C above l is maximum.

Solution. (see Fig. 1.25). If $AD = x$, then $DB^2 = c^2 - x^2$ by Pythagoras theorem. It is easily seen that $A\hat{B}D = B\hat{C}E$, $D\hat{A}B = C\hat{B}E$ so that $\Delta ABD|||\Delta BCE$.

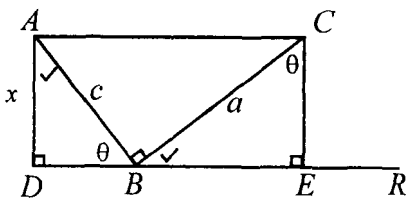


Fig. 1.25

Hence $\frac{AB}{BD} = \frac{BC}{CE}$. i.e., $\frac{c}{BD} = \frac{a}{CE}$

or $CE = \frac{a}{c}BD = \frac{a}{c}\sqrt{c^2 - x^2}$.

We are to maximize $x + \frac{a}{c}\sqrt{c^2 - x^2} = r$, say.

Then $(r - x)^2 = \frac{a^2}{c^2}(c^2 - x^2)$. $r^2 - 2xr + x^2 = \frac{a^2}{c^2}(c^2 - x^2)$;

i.e.,

$$r^2 = a^2 - \left(1 + \frac{a^2}{c^2}\right)x^2 + 2xr = a^2 - \left(\frac{c^2 + a^2}{c^2}\right)\left[x - \frac{c^2 r}{c^2 + a^2}\right]^2 + \frac{c^2 + a^2}{c^2} \cdot \frac{c^4 r^2}{(c^2 + a^2)^2}$$

i.e., $r^2\left(1 - \frac{c^2}{c^2 + a^2}\right) = a^2 - \frac{c^2 + a^2}{c^2}\left(x - \frac{c^2 r}{c^2 + a^2}\right)^2$.

so that x is maximum when $x = \frac{c^2 r}{c^2 + a^2}$.

Then $x + \frac{a}{c}\sqrt{c^2 - x^2} = \frac{c^2 + a^2}{c^2}x$,

i.e., $\frac{a^2}{c^2}x = \frac{a}{c}\sqrt{c^2 - x^2}$ i.e., $\frac{a^2}{c^2}x^2 = c^2 - x^2$,

i.e., $\left(\frac{a^2}{c^2} + 1\right)x^2 = c^2$, i.e., $x^2 = \frac{c^4}{a^2 + c^2}$;

i.e., $x = \dfrac{c^2}{\sqrt{c^2+a^2}}$.

The maximum value of $AD + CE = \dfrac{c^2}{\sqrt{c^2+a^2}} + \dfrac{a}{c}\sqrt{c^2 - \dfrac{c^4}{c^2+a^2}}$

$= \dfrac{c^2}{\sqrt{c^2+a^2}} + \dfrac{a^2}{c}\dfrac{c}{\sqrt{c^2+a^2}} = \sqrt{a^2+c^2}$ = hypotenuse of ΔABC.

Problem 37. In Problem 36, to find the position of the triangle when the sum of the areas of the Δs ABD, CBE is maximum.

Solution. From similar triangles ABD, BCE, we have

$$\frac{AB}{AD} = \frac{BC}{BE} \text{ or } BE = \frac{BC}{AB} \cdot AD = \frac{a}{c}x.$$

Hence $\Delta ABD + \Delta CBE = \dfrac{1}{2}[BD \times AD + BE \times CE]$

$$= \frac{1}{2}\left[x \times \sqrt{c^2-x^2} + \frac{a}{c}x \times \frac{a}{c}\sqrt{c^2-x^2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{a^2}{c^2}\right] x\sqrt{c^2-x^2}.$$

The problem is one of maximizing $x\sqrt{c^2-x^2}$, i.e., $x^2(c^2-x^2)$. The maximum is reached when $x = \dfrac{c}{\sqrt{2}}$ (see solution of, for instance, of Problem 34). Then $BD = \sqrt{c^2 - \dfrac{c^2}{2}} = \dfrac{c}{\sqrt{2}}$ so that ABD is a right angle.

Problem 38. A string of given length is attached to one end of the diameter of a circular disc. It is passed over part of the circumference of the disc. The rest of the string is made in to a straight line ending on the diameter produced. To find the radius of the disc such that the area bounded by the straight part of the string, the boundary of the disc and the diameter produced is maximum.

Solution. Let AD be the diameter of the circular disc centre C, let ABE be the string and let B be the extreme point of the string on the disc (Fig. 1.26).

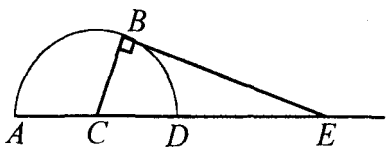


Fig. 1.26

Let the length of the string be l, constant. Let the radius of the disc, viz., $CB = x$, a variable. The area of the circular sector $ACB = \frac{1}{2}(\text{arc}AB) \times BC$.

Area of the semicircle $= \frac{\pi x^2}{2}$. Area BDE = sector $ACB + \Delta CBE -$ semicircle $= \frac{1}{2}[\text{arc}AB \times x + BE \times x] - \frac{\pi x^2}{2}$ $= \frac{1}{2}[x \times (\text{arc } AB + BE)] - \frac{\pi x^2}{2} = \frac{1}{2}(x \times l) - \frac{\pi x^2}{2} = \frac{1}{2}x(l - \pi x)$ so that the area is maximum when $x = \frac{l}{2\pi}$ (see solution of Problem 2 for instance).

Note: BE is tangent to the circle at B since the string is not fixed at B.

Problem 39. A is a point on the bounding diameter DB of a semicircle. To find a point E on the semicircle such that AE makes the smallest angle with the tangent at E to the semicircle.

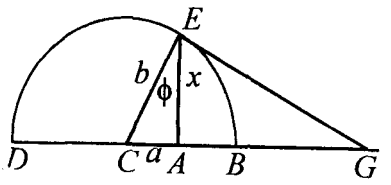


Fig. 1.27

Solution. As in Fig. 1.27 let the tangent to the circle centre C at E meet the diameter DB produced at G.

Let $CA = a, CE = b$, constants, $AE = x$,

$C\hat{E}A = \phi$, variables. By cosine formula applied to ΔCEA, $\cos\phi = \dfrac{b^2 + x^2 - a^2}{2bx}$. Note that ϕ is at the most a right angle since it is always less than $D\hat{E}B$. Thus when $A\hat{E}G$ is minimum, $\phi = C\hat{E}A$ is maximum and so $\cos\phi$ is minimum. i.e., $\dfrac{b^2 + x^2 - a^2}{2bx} = r$, a minimum. i.e., $b^2 + x^2 - a^2 = 2bxr$, where r is minimum. i.e., $x^2 - 2bxr = a^2 - b^2$, i.e., $(x - br)^2 = b^2r^2 + a^2 - b^2$ or $-b^2r^2 = (a^2 - b^2) - (x - br)^2$. Hence $-b^2r^2$ is maximum when $x = br$ or r is minimum when $x = br$. Then

$$b^2 + x^2 - a^2 = 2x^2$$

i.e.,
$$x^2 = b^2 - a^2$$

i.e.,
$$x = \sqrt{b^2 - a^2}$$

It follows that $C\hat{A}E$ is a right angle when ϕ is maximum or $A\hat{E}G$ is minimum.

Problem 40. If P is a given point on the bounding diameter AB of a semicircle to find the point D on the semicircle such that $AD + DP$ is maximum.

Solution. Let $DE \perp AB$. Let $AE = x$, variable; $AC = a$, $CP = b$, constants (Fig. 1.28(a)). Then, by a known result, DE is the mean proportional between AE and EB, i.e., $DE^2 = x(2a - x)$. (This is derived from the similarity of Δs ADE and DBE). Hence $PD^2 = DE^2 + EP^2 = x(2a - x) + (b + a - x)^2 = (a + b)^2 - 2bx$. Also, $\Delta ABD \sim \Delta ADE$ (see Fig. 1.28(b)), so that

$$\frac{AD}{AE} = \frac{AB}{AD} \quad \text{or} \quad AD^2 = AE \cdot AB,$$

i.e., $AD^2 = 2ax$. Then, $AD + DP = \sqrt{2ax} + \sqrt{(a + b)^2 - 2bx}$

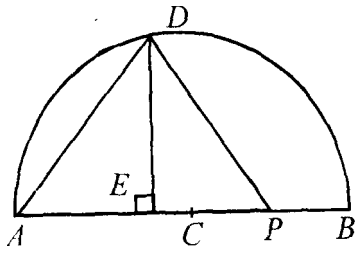


Fig. 1.28(a)

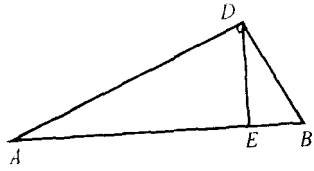


Fig. 1.28(b)

We are to maximize the above sum. Writing $\sqrt{2ax} = y$ we are to maximize $y + \sqrt{(a+b)^2 - \frac{b}{a}y^2}$. The maximum is reached (cf. solution of Problem 20) when $y = \dfrac{(a+b)}{\sqrt{\frac{b}{a}}\sqrt{\frac{b}{a}+1}} = \dfrac{a\sqrt{a+b}}{\sqrt{b}}$, if $b \neq 0$. So $x = \dfrac{y^2}{2a} = \dfrac{a^2(a+b)}{2ba} = \dfrac{a(a+b)}{2b}$.

Remarks. (1) The choice of x is such that $x : a = a+b : 2b$ or $2b : a+b = a : x$, i.e., $2CP : AP = AC : x$ i.e., x is the fourth proportional to $2CP$, AP, AC in that order according to the Indian tradition of the 'Rule of three'. Construct $AE = x$ and erect $ED \perp AB$ to meet the semicircle at D the desired point.

(2) Even if P lies on AB beyond B, the above solution holds (Fig. 1.28(c)).

(3) If P coincides with C, then for any D on the semicircle $AD+DP = AD+a$ and the maximum is reached when D coincides with B.

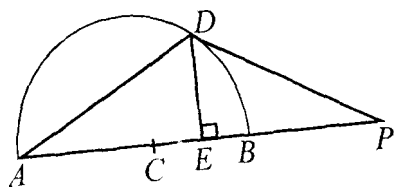


Fig. 1.28(c)

i.e., when $AD + DP = AB + a = 2a + a = 3a$. This is not reflected by the solution when we make $b \to 0$ in the solution of the problem.

(4) The maximum value for x possible is $2a$. This is reached when $\frac{a(a+b)}{2b} = 2a$ or $a + b = 4b$ or $b = \frac{a}{3}$, i.e., when CP is $\frac{1}{3}$ of the radius.

Problem 41. To find the right circular cone of the least volume which circumscribes a given sphere.

Solution. Let the vertical section of the cone and sphere through a vertical plane through the vertex of the cone be as in Fig. 1.29.

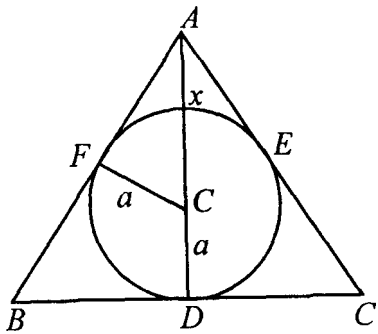


Fig. 1.29

The sections are a triangle and a circle inscribed therein respectively. If the radius of the sphere is a the radius of the incircle is a.

Let C be the centre of the circle (Note that the base of the cone should touch the sphere. Otherwise the cone will be larger than desired). Let $AC = x$, $BD = y$ the radius of the base of the cone, x, y being variables. Let the slant sides AB, AC of the cone, touch the sphere at F and E respectively. Then (since ACD is a straight line) $\Delta ACF \| \| \Delta ABD$ so that

$$\frac{AD}{BD} = \frac{AF}{CF} \quad \text{i.e.,} \quad \frac{x+a}{y} = \frac{\sqrt{x^2 - a^2}}{a}.$$

i.e., $$y = \frac{a(x+a)}{\sqrt{x^2 - a^2}}.$$

$$\text{Volume of the cone} = \frac{1}{3}\pi BD^2 \times AD$$
$$= \frac{1}{3}\pi y^2 \times (x+a) = \frac{1}{3}\pi \frac{a^2(x+a)^2}{x^2-a^2}(x+a)$$
$$= \frac{\pi a^2}{3}\frac{(x+a)^2}{x-a}$$

If $\frac{(x+a)^2}{x-a} = r$, we are to minimize r. Then

$$x^2 + 2ax + a^2 = rx - ar,$$

$$\text{i.e., } \left(x - \frac{r-2a}{2}\right)^2 = \frac{(r-2a)^2}{4} - ar - a^2 = \frac{r^2}{4} - 2ar$$
$$= \left(\frac{r}{2} - 2a\right)^2 - 4a^2.$$

So,
$\left(\frac{r}{2} - 2a\right)^2 = \left(x - \frac{r-2a}{2}\right)^2 + 4a^2$ and r is minimum when $x = \frac{r-2a}{2}$. i.e., $r = 2(x+a)$. Then $\frac{(x+a)^2}{x-a} = 2(x+a)$, i.e., $(x+a) = 2(x-a)$, if $x \neq a$, i.e., $x = 3a$. Hence the height of the cone is $x + a = 4a$, twice the diameter of the sphere.

Note: If the sphere is to be inscribed in the cone, note that $x > a, x \neq a$.

Problem 42. To find the positive real number such that its sum with its reciprocal is the least.

Solution. If $x > 0$, let $x + \frac{1}{x} = r$, i.e., $x^2 + 1 = rx$, i.e., $x^2 - rx + \frac{r^2}{4} = -1 + \frac{r^2}{4}$, i.e., $\frac{r^2}{4} = \left(x - \frac{r}{2}\right)^2 + 1$. Hence r is the least when $x = \frac{r}{2}$, i.e., $r = 2x$, i.e., $x + \frac{1}{x} = 2x$, i.e., $x^2 + 1 = 2x^2$, i.e., $x^2 = 1$. Hence $x = 1$.

Problem 43. AC, BD are given parallel straight line segments. To choose a point Y on BD such that if AD and CY meet at X, then the sum of the areas of Δs ACX, DXY is minimum (Fig. 1.30).

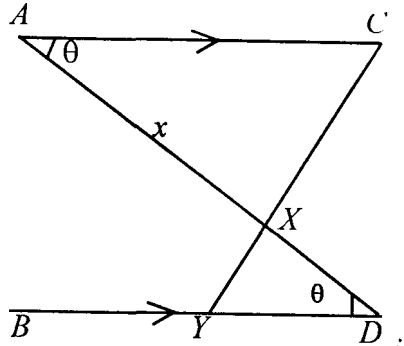


Fig. 1.30

Solution. Let $AC = a$, $AD = b$, and let by intercept theorem $A\hat{D}B = D\hat{A}C = \theta$, constants and let $AX = x$, varying with the choice of Y on BD. $\Delta AXC = \frac{1}{2}AX \cdot AC \sin\theta = \frac{1}{2}ax\sin\theta$ and $\Delta XDY = \frac{1}{2}DX \cdot DY\sin\theta = \frac{1}{2}(b-x)\cdot DY\sin\theta$. $\Delta ACX|||DYX$. So $\frac{DY}{a} = \frac{DY}{AC} = \frac{DX}{AX} = \frac{b-x}{x}$ so that $DY = \frac{a(b-x)}{x}$. Hence $\Delta XDY = \frac{1}{2}(b-x)\frac{a(b-x)}{x}\sin\theta = \frac{1}{2}a\frac{(b-x)^2}{x}\sin\theta$. So,

$$\Delta AXC + \Delta XDY = \frac{1}{2}a\sin\theta\left[x + \frac{(b-x)^2}{x}\right].$$

It is enough to minimize $x + \frac{(b-x)^2}{x} = r$.

Then $x^2 + b^2 + x^2 - 2bx = xr$, i.e., $2x^2 - x(r+2b) + b^2 = 0$ i.e., $\left(\sqrt{2}x - \frac{r+2b}{2\sqrt{2}}\right)^2 + b^2 = \frac{(r+2b)^2}{8}$. r is therefore minimum when $\sqrt{2}x = \frac{r+2b}{2\sqrt{2}}$, i.e., $4x = r + 2b$,

i.e., $r = 4x - 2b = 2(2x-b)$. So, $x + \frac{(b-x)^2}{x} = 2(2x-b)$.

i.e., $x^2 + (b-x)^2 = 4x^2 - 2bx$, i.e., $2x^2 = b^2$, i.e., $x^2 = \frac{b^2}{2}$,

i.e., $x = \dfrac{b}{\sqrt{2}}$.

Problem 44. An inelastic ball is dropped from a point C above a point A vertically downwards and impinges on the horizontal floor at B and bounces. To find the position of C so that the ball gets back to A at the least possible time.

Solution. The only force acting on the ball is the gravity g and hence the distance covered by the ball from time of dropping from rest to time t is given[†] $s = gt^2$. If the ball arrives at A from C at time t and at B at time t', then AB is covered during downward fall during the time $t' - t = \dfrac{\sqrt{CB}}{\sqrt{g}} - \dfrac{\sqrt{CA}}{\sqrt{g}} = \dfrac{1}{\sqrt{g}}\left(\sqrt{a+x} - \sqrt{x}\right)$ if $CA = x$ (variable), $AB = a$ (fixed).

Time taken to reach A after bouncing at B is $\frac{1}{\sqrt{g}}\sqrt{a+x}$. The time taken by the ball to reach A for a second time is $\frac{2}{\sqrt{g}}\sqrt{a+x} - \frac{1}{\sqrt{g}}\sqrt{x}$. The problem is to minimize $2\sqrt{a+x} - \sqrt{x} = r$, say; i.e., $2\sqrt{a+x} = r + \sqrt{x}$. Squaring $4(a+x) = r^2 + 2r\sqrt{x} + x$. $r^2 = 4a + 3x - 2r\sqrt{x} = 4a + 3\left(x - \frac{2r}{3}\sqrt{x} + \frac{r^2}{9}\right) - \frac{r^2}{3}$;

i.e., $\frac{4}{3}r^2 = 4a + 3(x - \frac{r}{3})^2$ so that r is minimum when $\sqrt{x} = \frac{r}{3}$,

i.e., $2\sqrt{a+x} - \sqrt{x} = 3\sqrt{x}$ i.e., $\sqrt{a+x} = 2\sqrt{x}$

i.e., $a + x = 4x$ i.e., $x = \frac{a}{3}$.

Problem 45. A weight w lies at the bottom of an inclined plane with prescribed height a of its highest edge from the horizontal. A given force equivalent to a mass of p in weight is applied to the weight in the direction of the inclined plane.

[†]This derivation depends on calculus using Newton's laws of motion.

To find the length of the inclined plane so that the weight w is pulled up to the upper edge of the plane at the least time.

Solution. The forces acting on w are (see Fig. 1.31) its weight and the force pg along the inclined plane. i.e., $wg\sin\alpha$ down the plane and pg up the plane if α is the angle of inclination of the plane. i.e., $pg - wg\sin\alpha$.

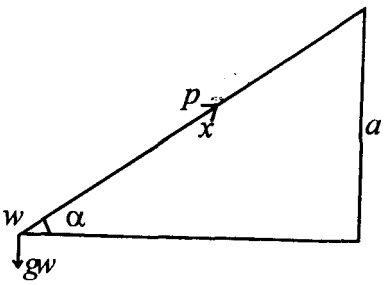


Fig. 1.31

By Newton's law the acceleration of w along the plane is $\frac{\text{Force}}{\text{mass}} = \frac{(pg - wg\sin\alpha)}{w}$. Now, $\sin\alpha = \frac{a}{x}$ if x is the length of the inclined plane. Thus the acceleration is $\frac{1}{w}(pg - wg\frac{a}{x}) = \frac{g}{w}\left(\frac{px - aw}{x}\right)$. If t is the time taken by w to reach the upper edge of the inclined plane, then x = acceleration $\cdot t^2$ so that $t^2 = x\Big/\frac{g}{w}\left(\frac{px - aw}{x}\right) = \frac{wx^2}{g(px - aw)}$. We are therefore to minimize $\frac{x^2}{px - aw} = r$, say. Then $x^2 = pxr - awr$, i.e., $x^2 - pxr + \frac{p^2r^2}{4} = \frac{p^2r^2}{4} - awr = \frac{p^2r^2}{4} - awr + \frac{1}{p^2}a^2w^2 - \frac{1}{p^2}a^2w^2 = \left(\frac{pr}{2} - \frac{1}{p}aw\right)^2$. Hence r is minimum when $x = \frac{pr}{2}$, i.e., $r = \frac{2x}{p}$. Thus $\frac{x^2}{px - aw} = \frac{2x}{p}$, i.e., $px = 2(px - aw)$. i.e., $px = 2aw$, i.e., $x = \frac{2aw}{p}$. i.e., $p : w = 2a : x$, i.e., twice height of inclination: length of the inclined plane = force: weight of particle.

Problem 46. A vessel is always kept full of water by a tap. A hole is to be made at its side so that the spout of water emanating from it reaches the maximum distance on the ground. To find the height from the top of the vessel to the hole.

Solution. If g is gravity, D is the hole, $AD = x$, $AB = a$ (Fig. 1.32), B being the bottom of the vessel, the spout of water from D takes the shape of a parabolic arc DG. The horizontal velocity of the spout at D is $v = 2\sqrt{gx}$. The time t taken for the spout to reach from D to G on the horizontal is

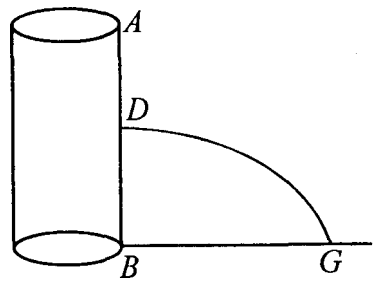


Fig. 1.32

given by $t^2 = \dfrac{DB}{\frac{1}{2}g} = \dfrac{a-x}{\frac{1}{2}g}$,† i.e., $t = \dfrac{\sqrt{2}\sqrt{a-x}}{\sqrt{g}}$.

So $vt = \dfrac{2\sqrt{gx}\sqrt{a-x}}{\sqrt{g}} = 2\sqrt{x(a-x)}$ is the horizontal distance GB.

Thus we are to maximize $x(a-x)$ and we know from an earlier solution that $x = \frac{a}{2}$ when $x(a-x)$ is maximum.

Note: The shape of the vessel is immaterial here.

†To be derived using the fact that pressure at a point in the liquid is equal to atmospheric pressure + $\rho g x$, ρ being the density of the liquid.

Problem 47. In Problem 46, to find the position of D if the bottom of the vessel is above ground level (see Fig. 1.33).

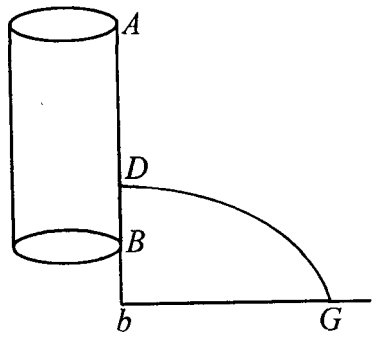


Fig. 1.33

Solution. Let $Ab = c$. As in the solution of Problem 46, $bG = 2\sqrt{x(c-x)}$ so that bG is maximum when $x = \frac{c}{2}$.

i.e., the hole should be made midway between the top of the vessel and the ground.

Problem 48. To express a given positive real number as the sum of two positive real numbers such that the difference between the product of the two summands and the square of one of them is maximum.

Solution. Let $a > 0$ be given and let the summands be $x, a - x$. To maximize $x(a - x) - x^2 = r$ say. Then $r = ax - 2x^2 = \frac{a^2}{8} - \left(\sqrt{2}x - \frac{a}{2\sqrt{2}}\right)^2$ so that r is maximum when $\sqrt{2}x = \frac{a}{2\sqrt{2}}$, i.e., $x = \frac{a}{4}$.

Note: If $(a-x)^2 < x(a-x)$, then we get $x(a-x)-(a-x)^2$ is maximum when $x = \dfrac{3a}{4}$. In this case we have $r = x(3a - 2x)$ or equivalently $2x(3a - 2x)$ so that the solution is $2x = \dfrac{3a}{2}$.

Problem 49. To find the point on the line of centres of two spheres from which the maximum spherical pairs of the two spheres is visible.

Solution. Let the centres of the spheres be A, B. Let the

radius of the spheres be AC, DB; $AC = a$, $DB = b$. Consider a plane section passing through the line of centres. Let one of the common transverse tangents QR cut the line of centres AB at P. Let L, M be the feet of perpendiculars on AB from the points Q, R of contact of the common tangent with the cross-section circles (see Fig. 1.34).

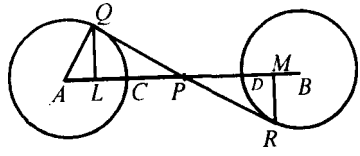


Fig. 1.34

Let $AB = c$. $\Delta AQP|||\Delta ALQ$ so that if $AP = x$

$$\frac{AQ}{AL} = \frac{AP}{AQ}, \quad \text{i.e.,} \quad \frac{a}{AL} = \frac{x}{a}.$$

Hence $AL = \dfrac{a^2}{x}$. $LC = AC - AL = a - \dfrac{a^2}{x}$. Similarly, $MD = b - \dfrac{b^2}{c - x}$. The surface of the part of the sphere centre A bound by the circle centre L and radius LQ is $2\pi a \times LC = 2\pi a\left(a - \dfrac{a^2}{x}\right)$. The surfaces of the two spheres visible from P have area equal to

$$2\pi a\left(a - \frac{a^2}{x}\right) + 2\pi b\left(b - \frac{b^2}{(c-x)}\right)$$

$$= 2\pi(a^2 + b^2) - 2\pi\left(\frac{a^3}{x} + \frac{b^3}{c-x}\right).$$

Thus we are to maximize this w.r.t. x or to minimize

$$\frac{a^3}{x} + \frac{b^3}{c-x} = \frac{ca^3 + (b^3 - a^3)x}{x(c-x)} = r, \quad \text{say.}$$

Then $ca^3 + (b^3 - a^3)x = rx(c - x)$,

i.e., $ca^3 = -(b^3 - a^3 - cr)x - rx^2 =$

$$\left(\frac{b^3 - a^3 - cr}{2\sqrt{r}}\right)^2 - \left(\sqrt{r}x + \frac{b^3 - a^3 - cr}{2\sqrt{r}}\right)^2$$

or $\left(\dfrac{b^3-a^3-cr}{2\sqrt{r}}\right)^2 = ca^3 + \left(\sqrt{r}x + \dfrac{b^3-a^3-cr}{2\sqrt{r}}\right)^2$

i.e., $\left(\dfrac{c}{2}\sqrt{r} - \dfrac{b^3-a^3}{2\sqrt{r}}\right)^2 = ca^3 + \left(\sqrt{r}x + \dfrac{b^3-a^3-cr}{2\sqrt{r}}\right)^2$.

If $\dfrac{c}{2}\sqrt{r} - \dfrac{b^3-a^3}{2\sqrt{r}}$ is minimum so is r since

$a_1\sqrt{r} - \dfrac{b_1}{\sqrt{r}} \le a_1\sqrt{r'} - \dfrac{b_1}{\sqrt{r'}} \Rightarrow r < r'$ when $a_1,\ b_1 \ge 0$.

Thus r is minimum when $\sqrt{r}x + \dfrac{b^3-a^3-cr}{2\sqrt{r}} = 0$,

i.e., $2rx + b^3 - a^3 - cr = 0$, i.e., $2rx + b^3 - a^3 - cr = 0$,

i.e., $r = \dfrac{b^3-a^3}{c-2x}$. Then $\dfrac{ca^3 + (b^3-a^3)x}{x(c-x)} = \dfrac{b^3-a^3}{c-2x}$.

So, $c^2a^3 + c(b^3-a^3)x - 2ca^3x - 2(b^3-a^3)x^2 = (b^3-a^3)cx -$

$(b^3-a^3)x^2$. i.e., $(b^3-a^3)x^2 + 2ca^3x - c^2a^3 = 0$;

i.e., $\left\{\sqrt{(b^3-a^3)}x + \dfrac{ca^3}{\sqrt{b^3-a^3}}\right\}^2$

$$= c^2a^3 + \frac{c^2a^6}{b^3-a^3} = \frac{c^2a^3\left[b^3-a^3+a^3\right]}{b^3-a^3} = \frac{c^2a^3b^3}{b^3-a^3}.$$

So, $x = \left[\pm\left\{\dfrac{c^2a^3b^3}{b^3-a^3}\right\}^{\frac{1}{2}} - \dfrac{ca^3}{\sqrt{b^3-a^3}}\right]\dfrac{1}{\sqrt{b^3-a^3}}$

$$= \frac{ca^{\frac{3}{2}}b^{\frac{3}{2}} - ca^3}{\sqrt{b^3-a^3}}\frac{1}{\sqrt{b^3-a^3}} = \frac{ca^{\frac{3}{2}}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)}{(b^3-a^3)} = \frac{ca^{\frac{3}{2}}}{\left(b^{\frac{3}{2}} + a^{\frac{3}{2}}\right)}.$$

Note: It has been assumed that $b > a$ and that P is distant from the sphere with radius a by less than half the distance between the centres of the two spheres.

Problem 50. To find x such that $\sin(x-\alpha)\cos x$ is maximum α being fixed.

Solution. $\cos x \sin(x-\alpha) = \cos x \sin x \cos\alpha - \cos^2 x \sin\alpha$. Let $\cos\alpha = a$, constant, $\cos x = y$, variable. $\sin\alpha = \sqrt{1-a^2}$. (assuming that α as also x is an acute angle to begin with). Then $\cos x \cdot \sin(x-\alpha)$

$$= y\sqrt{1-y^2}\cdot a - y^2\sqrt{1-a^2} = \sqrt{1-a^2}\left[\frac{a}{\sqrt{1-a^2}}\sqrt{y^2-y^4} - y^2\right]$$

so that it suffices to maximize $\dfrac{a}{\sqrt{1-a^2}}\sqrt{y^2-y^4} - y^2 = r$, say. Then

$$\frac{a^2}{1-a^2}(y^2-y^4) = (y^2+r)^2 = y^4 + 2y^2r + r^2;$$

i.e., $$\left(\frac{a^2}{1-a^2} - 2r\right)y^2 - \left(\frac{a^2}{1-a^2}+1\right)y^4 = r^2;$$

i.e., $$\left(\frac{a^2}{1-a^2} - 2r\right)y^2 - \frac{1}{1-a^2}y^4 = r^2;$$

i.e., $$\left(a^2 - 2r(1-a^2)\right)y^2 - y^4 = (1-a^2)r^2$$ (as we can assume $a \neq 1$); i.e.,

$$\left[\frac{a^2-2r(1-a^2)}{2}\right]^2 - \left(y^2 - \frac{a^2-2r(1-a^2)}{2}\right)^2 = (1-a^2)r^2.$$

r is therefore maximum when $y^2 = \dfrac{a^2-2r(1-a^2)}{2}$

i.e., $r = \dfrac{a^2-2y^2}{2(1-a^2)}$.

Then $$\frac{a}{\sqrt{1-a^2}}y\sqrt{1-y^2} - y^2 = \frac{a^2-2y^2}{2(1-a^2)}$$

i.e., $$\frac{a}{\sqrt{1-a^2}}y\sqrt{1-y^2} = \left(\frac{a^2-2y^2}{2(1-a^2)} + y^2\right)$$

$$= \frac{a^2-2y^2+2y^2-2a^2y^2}{2(1-a^2)}.$$

Squaring $\dfrac{a^2}{1-a^2}y^2(1-y^2) = \dfrac{a^4(1-2y^2)^2}{4(1-a^2)^2}$.

i.e., $4(1-a^2)y^2(1-y^2) = a^2(1-2y^2)^2$

i.e., $4y^2(1-a^2-y^2+a^2y^2) = a^2-4a^2y^2+4a^2y^4$

i.e., $4y^2-4a^2y^2-4y^4+4a^2y^4 = a^2-4a^2y^2+4a^2y^4$

i.e., $4y^4-4y^2+a^2 = 0,$

i.e., $y^2 = \dfrac{4\pm\sqrt{16-16a^2}}{8}$

$= \dfrac{1}{2}\left(1\pm\sqrt{1-a^2}\right)$

i.e., $y^2 = \dfrac{1}{2}(1\pm\sin\alpha);$

i.e., $y = \sqrt{\dfrac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\pm 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2}}$

$= \sqrt{\dfrac{\left(\cos\frac{\alpha}{2}\pm\sin\frac{\alpha}{2}\right)^2}{2}}$

$= \dfrac{1}{\sqrt{2}}\cos\dfrac{\alpha}{2}\pm\dfrac{1}{\sqrt{2}}\sin\dfrac{\alpha}{2} = \cos\left(\dfrac{\pi}{4}\mp\dfrac{\alpha}{2}\right) = \cos x$

So $x = \dfrac{\pi}{4}\mp\dfrac{\alpha}{2}$.

Remarks. (i) When $\alpha = 0$ the function becomes $\sin x\cos x = \dfrac{\sin 2x}{2}$ and the maximum value is attained when $2x = \dfrac{\pi}{2}$ or $x = \dfrac{\pi}{4}$.

(ii) When $0 \le x \le \dfrac{\pi}{2}$ and $\frac{\pi}{2} \le \alpha \le \pi$, then $\sin(x-\alpha)\cos x$ is non-positive and the maximum value is 0 and the minimum

value is the $-\max[-\sin(x-\alpha)\cos x]$. Now,

$$\begin{aligned} -\sin(x-\alpha)\cos x &= -\sin\left(x-(\alpha-\frac{\pi}{2})-\frac{\pi}{2}\right)\cos x \\ &= \cos\left(x-\left(\alpha-\frac{\pi}{2}\right)\right)\cos x \\ &= \cos(x-\beta)\cos x \\ &= \cos^2 x\cos\beta+\sin\beta\sin x\cos x \\ &= y^2\cos\beta+y\sqrt{1-y^2}\sin\beta \end{aligned}$$

if $y=\cos x$ and $\beta=\alpha-\frac{\pi}{2}$. Writing $\cos\beta=a, \sin\beta=\sqrt{1-a^2}$. We have to maximize $ay^2+y\sqrt{1-y^2}\sqrt{1-a^2}=r$;

i.e., $y\sqrt{1-y^2}\sqrt{1-a^2}=r-ay^2$.

Squaring, $(1-a^2)y^2(1-y^2)=r^2-2ray^2+a^2y^4$;

i.e., $(1-a^2)y^2-(1-a^2)y^4=r^2-2ray^2+a^2y^4$;

i.e., $(1+2ra-a^2)y^2-y^4=r^2$;

i.e., $\left(\frac{1+2ra-a^2}{2}\right)^2-\left(y^2-\frac{1+2ra-a^2}{2}\right)^2=r^2$.

r^2 is thus maximum when $y^2=\frac{1+2ra-a^2}{2}$;

i.e., $r=\frac{2y^2+a^2-1}{2a}$. Hence

$$\sqrt{1-a^2}y\sqrt{1-y^2}=\frac{2y^2+a^2-1}{2a}-ay^2.$$

Squaring, we get $4y^4-4y^2+(1-a^2)=0$ so that

$y^2=\frac{1}{2}(1\pm a)$, $y=\frac{1}{\sqrt{2}}\sqrt{1\pm a}=\frac{1}{\sqrt{2}}\sqrt{1\pm\cos\beta}=\frac{1}{\sqrt{2}}\sqrt{2}\cos\frac{\beta}{2}$ or $\frac{1}{\sqrt{2}}\sqrt{2}\sin\frac{\beta}{2}$ i.e., $y=\cos\frac{\beta}{2}$ or $\sin\frac{\beta}{2}$ i.e., $\cos x=\cos\frac{\beta}{2}$ or $\sin\frac{\beta}{2}=\cos\left(\frac{\pi}{2}-\frac{\beta}{2}\right)$.

Hence $x = \frac{\beta}{2} = \frac{\alpha}{2} - \frac{\pi}{4}$ or $\frac{\pi}{4} - \frac{\alpha}{2}$.

(iii) Ranges for x and α may be varied and the maximum or minimum as the case may be discussed accordingly.

Problem 51. To find x when $\frac{x(a-x)}{a^2}$ is maximum, when a is a positive constant. (If a is negative $\frac{x(a-x)}{a^2}$ becomes large when x becomes negatively large. If $a > 0$, for $x > a$, the function $\frac{x(a-x)}{a^2}$ becomes negatively large for large x. If $0 < x < a$, $\frac{x(a-x)}{a^2}$ is positive. For $x < 0$, the function is negative and becomes negatively large for negatively large x. Thus the interesting case is when $a > 0$ and $0 \leq x \leq a$).

Solution. $\frac{1}{a^2}$ being positive, $x(a-x)$ is the product of two positive real numbers whose sum is always a. The maximum occurs when $x = \frac{a}{2}$ (see Problem 1, Chapter I) and the maximum is $\frac{a}{2} \cdot \frac{(a - \frac{a}{2})}{a^2} = \frac{1}{4}$.

Problem 52. A regular hexagonal prism is placed vertically and a tetrahedron with its base being the triangle with alternate vertices of the regular hexagonal face on the top as vertices and with its vertex lying on the axis of the regular hexagonal prism is placed above the prism and the parts of the prism extending beyond the faces of the tetrahedron are cut off. Find the inclination of the faces of the tetrahedron to the axis of the prism if the surface of the solid is least with prescribed volume for the solid equal to that of the prism.

Solution. As in Fig. 1.35, let $ABCDEF$ be the upper face of the prism which is a regular hexagon. Let the base of the tetrahedron be BDF and S its vertex. Then SO

is perpendicular to the plane of the hexagon. O being the centre of the circle which circumscribes the regular hexagon.

Let AO meet BF at M. Let $M\hat{S}O = \theta$. Let a be the side of the hexagon and h the height of the prism. $FOBA$ is a rhombus of side a and so FB and OA are perpendicular bisectors of each other and so $OM = MA = \frac{a}{2}$, since $OA = a$. So, $SM = \frac{a}{2}\text{cosec}\,\theta$ and $OS = \frac{a}{2}\cot\theta$. Each of the three portions cut off from the hexagonal prism is of the form of a tetrahedron with an isosceles triangle with equal sides of length a containing an angle of $120°$ and the portion of the edge of the prism cut off is perpendicular to the base. If h' is the height of the tetrahedron, since the volume of the tetrahedron put up on the top of the prism on the base of an equilateral triangle of side $\sqrt{3}a$. and height $\frac{a}{2}\cot\theta$ is equal to the sum of the volumes of the three tetrahedrons cut off, we have

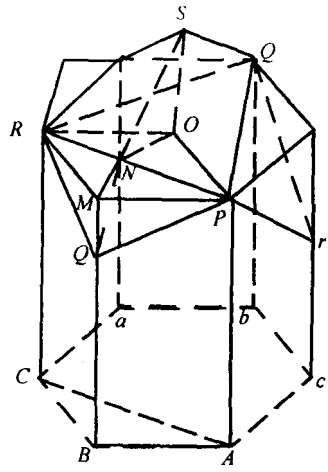


Fig. 1.35

$$\frac{\sqrt{3}}{4}3a^2 \times \frac{a}{2}\cot\theta = 3 \times \frac{1}{2}a^2\frac{\sqrt{3}}{2} \times h'$$

so that $h' = \frac{a}{2}\cot\theta$. Thus in the rectangle of a face of the prism of area ah an area equal to $\frac{1}{2}a \cdot \frac{a}{2}\cot\theta$ is omitted. Thus the surface area of the new solid in this part is $ah - \frac{1}{4}a^2\cot\theta$. The total surface except that of the

tetrahedron at the top is $6\left(ah - \frac{1}{4}a^2 \cot\theta\right)$. The surface of the tetrahedron at the top is made up of those triangles with base $\sqrt{3}a$ and height $SM = \frac{a}{2}$ cosec θ so that surface area is $\frac{3}{2}\sqrt{3}a \cdot \frac{a}{2}$ cosec $\theta = \frac{3\sqrt{3}}{4}a^2$ cosec θ.

The area of the triangular face got by slicing the prism at the top is the same as that of one face of the tetrahedron above it. The reason is that both have the same base and have an altitude which has the same projection of $\frac{a}{2}\cot\theta$ in the direction of the axis of the prism.

The total surface area of the new solid is thus $6\left(ah - \frac{1}{4}a^2 \cot\theta\right) + \frac{3\sqrt{3}}{2}\frac{a^2}{2}$ cosec θ. Omitting the constants we are to minimize $\sqrt{3}$ cosec $\theta - \cot\theta = r$, say. Putting $\cot\theta = x$, we have $\sqrt{3}\sqrt{1+x^2} - x = r$, i.e., $(r+x) = \sqrt{3}\sqrt{1+x^2}$. Squaring $r^2 + 2xr + x^2 = 3 + 3x^2$, $2x^2 - 2xr = r^2 - 3$, i.e., $\left(\sqrt{2}x - \frac{r}{\sqrt{2}}\right)^2 = r^2 - \frac{r^2}{2} - 3 = \frac{r^2}{2} - 3$. Thus r is minimum when $\sqrt{2}x = \frac{r}{\sqrt{2}}$ or $r = 2x$. Then $\sqrt{3}\sqrt{1+x^2} - x = 2x$, i.e., $\sqrt{3}\sqrt{(1+x^2)} = 3x$, i.e., $\sqrt{1+x^2} = \sqrt{3}x$, i.e., $1 + x^2 = 3x^2$, i.e., $2x^2 = 1$ or $x^2 = \frac{1}{2}$, i.e., $x = \frac{1}{\sqrt{2}}$. i.e., $\cot\theta = \frac{1}{\sqrt{2}}$ or $\tan\theta = \sqrt{2}$.

Remark. The cells of the bee hive have the shape described above.

Problem 53. To find x for which $\frac{x}{(a+x)(b+x)}$ is maximum $a, b > 0$ being constants.

Solution.

To minimize $\dfrac{(a+x)(b+x)}{x}$, i.e., $\dfrac{ab+(a+b)x+x^2}{x}$, i.e., $(a+b)+\dfrac{ab}{x}+x$. It is enough to minimize $\dfrac{ab}{x}+x$ for $x>0$.

Clearly $\dfrac{ab}{x}+x=\sqrt{ab}\left(\dfrac{\sqrt{ab}}{x}+\dfrac{x}{\sqrt{ab}}\right)$ so that by solution to Problem 42, the minimum is reached when $\dfrac{x}{\sqrt{ab}}=1$ or $x=\sqrt{ab}$ and the least value is $2\sqrt{ab}$.

Problem 54. Given the difference of two positive real numbers to find the numbers such that the third proportional of the lesser to the greater is the least.

Solution: If $a>0$ is the difference and x is the greater number, the lesser number is $x-a$. The third proportional is y such that $(x-a):x=x:y$. Thus $y=\dfrac{x^2}{x-a}=\dfrac{x^2-a^2+a^2}{x-a}=x+a+\dfrac{a^2}{x-a}=2a+x-a+\dfrac{a^2}{x-a}$.

It suffices to minimize $x-a+\dfrac{a^2}{x-a}$ or $x'+\dfrac{a^2}{x'}$. From the solution to Problem 53, $x'=\sqrt{a^2}=a$. i.e., $x-a=a$, i.e., $x=2a$. The minimum value of the third proportional is $\dfrac{4a^2}{2a-a}=4a$.

Problem 55. To find the dimension of a right circular cone of the least surface area for a given volume.

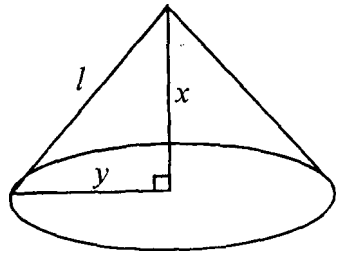


Fig. 1.36

Solution. Let x be the height of the cone and y the radius of the base (see Fig.1.36). Then the volume is $\dfrac{1}{3}\pi y^2 x=\dfrac{1}{3}\pi a^3$, say where a

is constant. Surface area = area of slant surface + area of base = area of sector of radius equal to slant height and perimeter equal to the circumference of the base + $\pi y^2 = \frac{2\pi y}{2\pi\sqrt{x^2+y^2}} \cdot \pi(x^2+y^2) + \pi y^2$ (since the slant height is $\sqrt{x^2+y^2}$) $= \pi\left\{y\sqrt{(x^2+y^2)}+y^2\right\}$. But $y^2 = \frac{a^3}{x}$ so that the surface area is $\pi\left\{\frac{a^{3/2}\sqrt{a^3+x^3}}{x}+\frac{a^3}{x}\right\}$. We are to minimize $\frac{\sqrt{a^3+x^3}+a^{3/2}}{x} = r$, say. $\sqrt{a^3+x^3} = rx - a^{3/2}$ or $a^3 + x^3 = (rx - a^{3/2})^2 = r^2x^2 - 2ra^{3/2}x + a^3$, i.e., $x^3 = r^2x^2 - 2ra^{3/2}x$, i.e., $x^2 = r^2x - 2ra^{3/2}$ since $x \neq 0$; so $\left(x-\frac{r^2}{2}\right)^2 = \frac{r^4}{4} - 2ra^{3/2}$. Thus $\frac{r^4}{4} = 2ra^{3/2} + \left(x-\frac{r^2}{2}\right)$. Hence r is minimum when $x = \frac{r^2}{2}$ when $\frac{r^4}{4} = 2ra^{3/2}$, i.e., $r^3 = 8a^{3/2}$, i.e., $r = 2a^{1/2}$ and $x = 2a$.

CHAPTER II

Problems in Maximum and Minimum in the Solution of which Cubic Equations are used

Problem 1. Find $x > 0$ such that when x^3 is subtracted from x, the remainder is maximum.

Solution: To find $x > 0$ such that $x - x^3$ is maximum. $x - x^3 = x(1 - x^2) \leq 0$ if $x \geq 1$; so, we can assume $0 < x < 1$. Write $x - x^3 = r$ or $x^3 - x = -r$; i.e., $x^3 - x + r = 0$. There is at least one real root for this equation. If $\alpha, \gamma + i\delta, \gamma - i\delta, \delta \geq 0$ (Note that complex roots occur in conjugate pairs.) are the roots, their product, viz., $\alpha(\gamma + i\delta)(\gamma - i\delta) = \alpha(\gamma^2 + \delta^2) = -r < 0$. So, $\alpha < 0$ (Note that r takes strictly positive values for $0 < x < 1$). Write $\alpha = -a, a > 0$. Then $x^3 - x + r = (x + a)(x^2 - ax + a^2 - 1)$ (as seen by long division noting that $x + a$ is a factor of the l.h.s.) and $r = a^3 - a$. Thus $\frac{r}{a} = a^2 - 1$ so that

$$x^3 - x + r = (x+a)(x^2 - ax + \frac{r}{a}) = (x+a)\left\{(x - \frac{a}{2})^2 + \frac{r}{a} - \frac{a^2}{4}\right\}.$$

Now, the x for which r is maximum should also satisfy the equation

$$\left(x - \frac{a}{2}\right)^2 + \frac{r}{a} - \frac{a^2}{4} = 0$$

or

$$\frac{r}{a} = \frac{a^2}{4} - (x - \frac{a}{2})^2.$$

So, r is maximum when $x = \frac{a}{2}$ and $r = \frac{a^3}{4}$. Hence for this value of x,

$$\frac{a}{2} - \frac{a^2}{4} = \frac{a^3}{4},$$

i.e., $a = 0$ or $\frac{3}{4}a^2 = 1$, i.e., $a = \frac{2}{\sqrt{3}}$

i.e., $x = \frac{a}{2} = \frac{2}{\sqrt{3}} \times \frac{1}{2} = \frac{1}{\sqrt{3}}$.

Problem 2. Find $x > 0$ such that when its cube is subtracted from its square, the remainder is maximum.

Solution: The problem is to maximize $x^2 - x^3$ and it suffices (cf. solution of Problem 1 in this chapter) to consider $x, 0 < x < 1$. Let $x^2 - x^3 = r(> 0)$ or $x^3 - x^2 = -r$, i.e., $x^3 - x^2 + r = 0$. As in the solution of Problem 1, this equation has a negative root, say $-a, a > 0$. Then $x^3 - x^2 + r = (x + a)[x^2 - (a + 1)x + a^2 + a]$ and the remainder $r - a(a^2 + a) = 0$. i.e., $r = a(a^2 + a)$. Then $a^2 + a = \frac{r}{a}$. Hence for maximum r, the corresponding x satisfies the quadratic equation $x^2 - (a+1)x + \frac{r}{a} = 0$. So,

$$\frac{r}{a} = \frac{(a+1)^2}{4} - \left(x - \frac{a+1}{2}\right)^2$$

so that r is maximum when

$$x = \frac{a+1}{2}, \quad \frac{r}{a} = \frac{(a+1)^2}{4}.$$

Then

$$\left(\frac{a+1}{2}\right)^2 - \left(\frac{a+1}{2}\right)^3 = \frac{a(a+1)^2}{4}.$$

Since $a + 1 \neq 0, 2 - (a + 1) = 2a$, i.e., $3a = 1$ or $a = \frac{1}{3}$,

so $x = \dfrac{\frac{1}{3}+1}{2} = \dfrac{2}{3}$

$$r = \frac{a(a+1)^2}{4} = \frac{\frac{1}{3}\left(\frac{1}{3}+1\right)^2}{4} = \frac{4^2}{3 \times 3^2 \times 4} = \frac{4}{3^3} = \frac{4}{27}$$

Problem 3. To find the isosceles triangle of the least area which can circumscribe a given circle.

Solution: Let O be the centre of the given circle of radius a and let ABC be an isosceles triangle circumscribing the circle with $AB = AC$. Let D be the point of contact of AC with the circle. Let $AO = x$ (variable) (see Fig. 2.1).

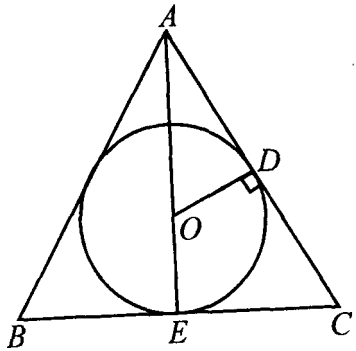


Fig. 2.1

Note that AO is bisector of $B\hat{A}C$ and so is also the altitude through A so that AO extended to meet BC at E is perpendicular to BC. Thus OE is perpendicular to BC, E being the point of contact of the tangent BC with the circle.

Now $\Delta AOD \;|||\; \Delta ACE$ so that $\dfrac{AD}{OD} = \dfrac{AE}{CE}$, i.e., $\dfrac{\sqrt{x^2-a^2}}{a} = \dfrac{x+a}{CE}$ i.e., $CE = \dfrac{a(x+a)}{\sqrt{x^2-a^2}}$. So, area of $\Delta ABC = \dfrac{1}{2} \times 2CE \times AE = \dfrac{a(x+a)^2}{\sqrt{x^2-a^2}}$. Squaring, we are to find the minimum of $\dfrac{(x+a)^4}{x^2-a^2} = \dfrac{(x+a)^3}{x-a} = \dfrac{y^3}{y-2a}$ if $y =$

$x + a$. If $\dfrac{y^3}{y - 2a} = r$, $y^3 - ry + 2ar = 0$. One of the roots of this equations is negative, say $-b, b > 0$. Dividing by $y+b$, we have $y^3 - ry + 2ar = (y+b)(y^2 - by + b^2 - r)$ and the remainder $2ar - b(b^2 - r) = 0$, i.e., $b^3 - br = 2ar$. i.e., $r = \dfrac{b^3}{2a + b}$. Now, $y^2 - by + b^2 - r = 0$, i.e., $\left(y - \dfrac{b}{2}\right)^2 + \dfrac{3b^2}{4} = r$ so that r is minimum when $y = \dfrac{b}{2}$. Then $r = \dfrac{3b^2}{4} = \dfrac{b^3}{2a + b}$ i.e., $3(2a + b) = 4b$, i.e., $b = 6a, y = 3a$. i.e., $x + a = 3a$ or $x = 2a$, i.e., $AO : OE = 2 : 1$. Hence O is the centroid of ΔABC or ΔABC is equilateral. To construct ΔABC, take A on a radius of the circle centre O such that $OA =$ twice radius. Draw the pair of tangents to the circle from A. Extend AO to meet the circle at E and draw the tangent to the circle at E to meet the tangents from A at B and C. Then ΔABC is the required triangle.

Problem 4. To find the right circular cylinder of maximum volume which can be inscribed in the given right circular cone so that the bases of both are in the same plane.

Solution: The centre of the bases of both the cylinder and the cone should coincide say at C since the cylinder is inscribed in the cone (see Fig. 2.2). Let AC be a (radius of the base of the cone) and BC (height of the cone) be b. Let x be the

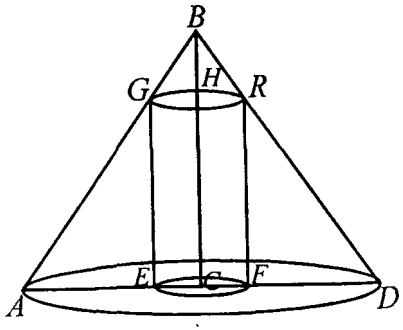


Fig. 2.2

diameter of the cylinder, considered as variable.

Let the altitude from the vertex of the cone to its base meet the top of the cylinder at H. Let GH be parallel to AD. ABD being a cross section of the cone.

$\Delta AEG \,|||\, \Delta ACB$ so that $\dfrac{AE}{AC} = \dfrac{EG}{CB}$, i.e., $\dfrac{a-x}{a} = \dfrac{EG}{b}$.

Hence $EG = \dfrac{b}{a}(a-x)$. So volume of cylinder $= \pi\dfrac{x^2}{4} \times EG = \dfrac{\pi x^2}{4}\dfrac{b}{a}(a-x) = \dfrac{\pi b}{4a}x^2(a-x)$. Thus, we are to maximize $x^2(a-x)$. i.e., to maximize $ax^2 - x^3$ or $\dfrac{x^2}{a^2} - \dfrac{x^3}{a^3}$; putting $\dfrac{x}{a} = y$, to maximize $y^2 - y^3$. From the solution to Problem 2 in this chapter the maximum is reached when $y = \dfrac{2}{3}$, i.e., $x = \dfrac{2}{3}a$. Also

$$EG = \frac{b}{a}(a-x) = \frac{b}{a}(a - \frac{2}{3}a) = \frac{b}{3}.$$

Problem 5. To find the dimensions of a cylindrical vessel of given capacity with the least internal surface area.

Solution: Let $ABCD$ be a vertical cross section of the vessel (see Fig. 2.3). Let $AB = x$ (diameter of base), $AD = y$ (height).

Internal surface of the cylinder D is $\pi xy + \pi\dfrac{x^2}{4}$. Volume of the cylinder is $\pi\dfrac{x^2 y}{4} = V$ (given) say. Then $y = \dfrac{4V}{\pi x^2}$.

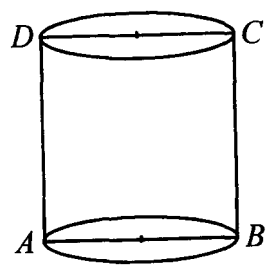


Fig. 2.3

Thus

$$\text{Surface area } = \pi x \ \frac{4V}{\pi x^2} + \pi\frac{x^2}{4} = \frac{4V}{x} + \frac{\pi x^2}{4} = r, \text{ say.}$$

We are to minimize r, $16V + \pi x^3 = 4rx$ or $\pi x^3 - 4rx + 16V = 0$ or $x^3 - \frac{4r}{\pi}x + \frac{16V}{\pi} = 0$. As in earlier cases, the above cubic has a negative real root, say $-a, a > 0$. Then

$$x^3 - \frac{4r}{\pi}x + \frac{16V}{\pi} = (x+a) \times \left(x^2 - ax + a^2 - \frac{4r}{\pi}\right)$$

and the remainder after dividing by $(x+a)$ being 0, $\frac{16V}{\pi} - a^2 + \frac{4ar}{\pi} = 0$ or $a^2 - \frac{4ar}{\pi} = \frac{16V}{\pi}$. Hence

$$r = \frac{\pi a^2 - 16V}{\pi} \times \frac{\pi}{4a} = \frac{\pi a^2 - 16V}{4a}.$$

x corresponding to minimum r satisfies

$$x^2 - ax + a^2 - \frac{4r}{\pi} = 0,$$

i.e.,
$$\left(x - \frac{a}{2}\right)^2 + \frac{3}{4}a^2 = \frac{4}{\pi}r.$$

Thus r is minimum when $x = \frac{a}{2}$ and $\frac{4}{\pi}r = \frac{3}{4}a^2$, i.e., $r = \frac{3\pi}{16}a^2$. Now, $16V + \pi\frac{a^3}{2^3} = 4\frac{3\pi}{16}a^2\frac{a}{2}$, i.e., $a^3\left[\frac{3\pi}{8} - \frac{\pi}{8}\right] = 16V$, i.e., $\frac{\pi}{4}a^3 = 16V$ or $a^3 = 64\frac{V}{\pi}$ or $a = 4\sqrt[3]{\frac{V}{\pi}}$ and so $x = \frac{a}{2} = 2\sqrt[3]{\frac{V}{\pi}}$. $r = \frac{3\pi}{16} \times 16\left(\frac{V}{\pi}\right)^{\frac{2}{3}} = 3\pi\left(\frac{V}{\pi}\right)^{\frac{2}{3}} = 3\pi^{\frac{1}{3}}V^{\frac{2}{3}}$.

Height $y = \frac{4V}{\pi x^2} = \frac{4V\pi^{\frac{2}{3}}}{4\pi V^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} = \sqrt[3]{\frac{V}{\pi}} =$ half of the diameter of the base.

Problem 6. To find the parabola which circumscribes a given circle and encompasses the smallest area between itself, the

circle and the tangents to the circle parallel to the tangent at the vertex to the parabola away from it.

Solution: Let P be one of the points of contact of the circle and the parabola (Fig. 2.4).

If C is the centre of the circle, then CP is normal to the parabola and is equal in length to another normal from C. Hence C is on the axis of the parabola. Let M be the foot of the perpendicular from P on to the axis of the parabola.

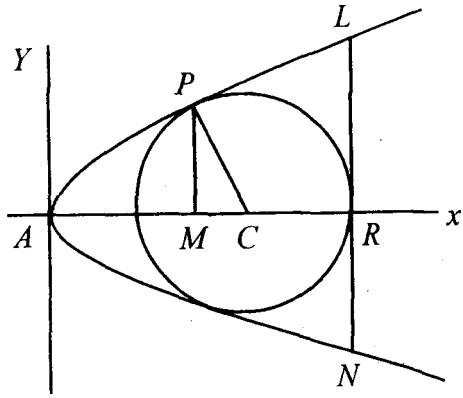


Fig. 2.4

Then CM is the subnormal from P, $CM = \frac{1}{2}$ the latus rectum $= z$, say. The equation of the parabola referred to its axis as x-axis and the tangent at its vertex A as y-axis is $y^2 = 2zx$. $PM^2 = a^2 - z^2$ if a is the radius of the circle. The area of the parabola bounded by the tangent at R (the other end of the diameter through A of the circle) is $\frac{4}{3}(AR)(LR)^*$. Now, $LR^2 = 2z(AR)$ (since L lies on the parabola). Thus the area $= \frac{4}{3}AR\sqrt{2z.AR}$. $AR = AM + MC + CR = \frac{a^2 - z^2}{2z} + z + a$ (since P lies on the parabola $PM^2 = 2z.AM.$).

Hence we are to minimize

$$z \cdot AR^3 = z\left[\frac{a^2 - z^2}{2z} + z + a\right]^3 = \frac{\left[a^2 - z^2 + 2z^2 + 2az\right]^3}{8z} = \frac{[z + a]^6}{8z^2}.$$

Thus, it is enough to minimize $\frac{(z+a)^3}{z}$. If $\frac{(z+a)^3}{z} = r$,[†] $(z+a)^3 - rz = 0$. Putting $y = z + a$, $y^3 - r(y-a) = 0$, i.e., $y^3 - ry + ar = 0$.

If $-\alpha, \alpha > 0$, is the negative real root of this equation, then $y^3 - ry + ar = (y+\alpha)(y^2 - \alpha y + \alpha^2 - r)$ and since the remainder when divided by $y + \alpha$ is zero $ar - \alpha^3 + \alpha r = 0$, i.e., $r = \frac{\alpha^3}{\alpha + a}$. For minimum r, y satisfies the quadratic equation $y^2 - \alpha y + \alpha^2 - r = 0$, i.e., $r = \left(y - \frac{\alpha}{2}\right)^2 + \frac{3}{4}\alpha^2$ so that r is minimum when $y = \frac{\alpha}{2}$ and then $r = \frac{3}{4}\alpha^2$. i.e., $\frac{3}{4}\alpha^2 = \frac{\alpha^3}{\alpha + a}$, i.e., $3(\alpha + a) = 4\alpha$, i.e., $\alpha = 3a$. So, $y = \frac{3a}{2}$. Thus $z = \frac{a}{2}$. The latus rectum of the parabola is a, viz., equal to the radius of the circle.

Problem 7. To find the dimension of the edges to be cut off from a rectangle sheet of length a and breadth b so that a box with maximum volume can be formed.

Solution: Assume edges of the form of a square of side x have been cut off. Then the volume of the box to be formed is $x(a-2x)(b-2x) = 4x^3 - 2(a+b)x^2 + abx$. It is enough to maximize this expression. Let $4x^3 - 2(a+b)x^2 + abx = 4r$. Then

$$x^3 - \frac{a+b}{2}x^2 + \frac{ab}{4}x - r = 0.$$

To remove the x^2-term put $x = y + \frac{a+b}{6}$. Then

[†]See Introduction, Remarks on (11).

$$x^3 = y^3 + \frac{a+b}{2}y^2 + \frac{(a+b)^2}{12}y + \frac{(a+b)^3}{6^3}$$

$$-\frac{a+b}{2}x^2 = -\frac{a+b}{2}y^2 - \frac{(a+b)^2}{6}y - \frac{(a+b)^3}{2\times 6^2}.$$

$$\frac{ab}{4}x = \frac{ab}{4}y + \frac{ab(a+b)}{4\times 6}.$$

so that the equation becomes

$$y^3 - \frac{(a+b)^2 - 3ab}{12}y - r = \frac{(a+b)^3}{72} - \frac{(a+b)^3}{216} - \frac{ab(a+b)}{24}.$$

The quantity on the r.h.s being constant r is maximum when

$$y^3 - \frac{(a+b)^2 - 3ab}{12}y = r' \text{ is maximum.}$$

Note that

$$\frac{(a+b)^3}{72} - \frac{(a+b)^3}{216} - \frac{ab(a+b)}{24} = \frac{2}{3\times 72}(a+b)^3 - \frac{ab(a+b)}{24}$$

$$= \frac{2(a+b)^3 - 9ab(a+b)}{3\times 72} = \frac{(a+b)[2a^2 - 5ab + 2b^2]}{3\times 72}$$

$$= \frac{(a+b)(2a-b)(a-2b)}{3\times 72} > 0.$$

if $a > 2b$.[‡] Under this condition $r' > 0$ and so the cubic $y^3 - \frac{(a+b)^2 - 3ab}{12}y - r' = 0$ has a positive real roots $\alpha > 0$.

Then

$$y^3 - \frac{(a+b)^2 - 3ab}{12}y - r' = (y-\alpha)(y^2 + \alpha y + \alpha^2 - n)$$

when $n = \frac{(a+b)^2 - 3ab}{12}$ and the remainder $-r' + (\alpha^3 - \alpha n) = 0$, i.e., $r' = \alpha^3 - \alpha n$. So y corresponding to maximum r satisfies $y^2 + \alpha y + \frac{r'}{\alpha} = 0$, i.e., $\frac{r'}{\alpha} = \frac{\alpha^2}{4} - \left(y + \frac{\alpha}{2}\right)^2$.

[‡]This restriction is found unnecessary when calculus is used.

Hence r' is maximum when $y = -\frac{\alpha}{2}$. Then $\frac{r'}{\alpha} = \frac{\alpha^2}{4}$, i.e., $r' = \frac{\alpha^3}{4}$. So $\frac{\alpha^3}{4} = \alpha^3 - \alpha n$, i.e., $\frac{3}{4}\alpha^3 = \alpha n$, i.e., $\frac{3}{4}\alpha^2 = n$, i.e., $\alpha = \frac{2\sqrt{n}}{\sqrt{3}}$. Hence

$$x = y + \frac{a+b}{6} = -\frac{\alpha}{2} + \frac{a+b}{6} = -\sqrt{\frac{n}{3}} + \frac{a+b}{6}$$
$$= \frac{a+b}{6} + \sqrt{\frac{(a+b)^2 - 3ab}{36}} = \frac{a+b+\sqrt{a^2-ab+b^2}}{6}.$$

Problem 8. To inscribe a rectangle of largest area in a parabolic sector bounded by a chord parallel to the tangent at the vertex.

Solution:
Clearly the rectangle should have one of its sides along the chord (Fig. 2.5).

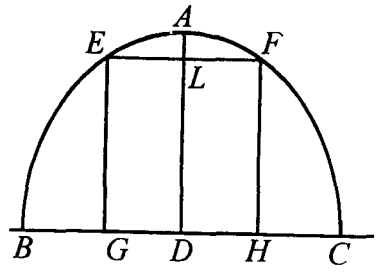


Fig. 2.5

Let A be the vertex of the parabola and BC the chord which is part of the boundary of the figure. Let AD be the axis of the parabola. The rectangle is then symmetrical about the axis of the parabola. Let the side EF meet the axis at L and let $AL = x$, variable. Let $AD = b$, a constant. Then $GE = DL = b - x$. If the parabola referred to its axis and its tangent at vertex as x, y-axis is $y^2 = 4ax$, then $EL = \sqrt{4ax} = 2\sqrt{ax}$. $EF = 2EL = 4\sqrt{ax}$.

$$\text{Area of } EFGH = 4\sqrt{ax} \times (b-x) = 4\sqrt{a}\sqrt{x}(b-x).$$

Thus we are to maximize $\sqrt{x}(b-x)$ or, equivalently, $b\sqrt{x} -$

$x^{\frac{3}{2}} = b^{\frac{3}{2}}\left(\sqrt{\frac{x}{b}} - \left(\frac{x}{b}\right)^{\frac{3}{2}}\right)$ or, equivalently, $\sqrt{\frac{x}{b}} - \left(\sqrt{\frac{x}{b}}\right)^3 = y - y^3$ where $y = \sqrt{\frac{x}{b}}$. From the solution to Problem 1 of this chapter, the maximum is reached when $y = \frac{1}{\sqrt{3}}$ or $\sqrt{\frac{x}{b}} = \sqrt{\frac{1}{3}}$ or $x = \frac{b}{3}$.

Note: The choice of AL depends only on AD and not on the shape of the parabola.

Problem 9. To find two summands of a given positive number such that their product multiplied by the difference of their squares is a maximum.

Solution Let the given number be $2a$ and the two summands $a - x$, $a + x$.

Then to maximize $(a-x)(a+x).4ax$, i.e., $4ax(a^2-x^2)$. It is enough to maximize $a^2x - x^3$, or equivalently $\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3$. As in the solution to Problem 8 above, the maximum is reached when $\frac{x}{a} = \frac{1}{\sqrt{3}}$ or $x = \frac{a}{\sqrt{3}}$.

Problem 10. To inscribe the ellipse of maximum area in a given isosceles triangle with one of its axis along the altitude through the vertex where the equal sides meet.

Solution: Let AD be the altitude of the isosceles triangle ABC (Fig. 2.6).

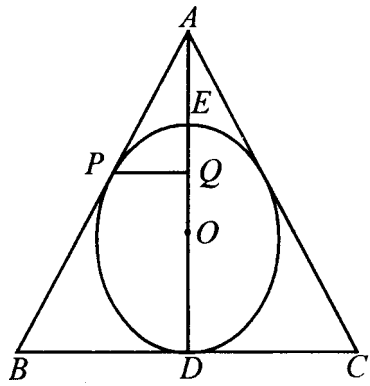


Fig. 2.6

The ellipse touches the side BC at D. Let it touch AB at P. Let O be the centre of the ellipse. O lies on AD.

Let E be the other extremity of the axis of the ellipse through D. Let $ED = 2x$ and let the other axis be of length $2y$. Let $AD = a$, $BD = b$, constants. Let Q be the foot of the perpendicular from P on AD. Then from the equation of the ellipse, the tangent at $P(OQ, PQ)$ passes through $A(x_0, 0)$ say.

So, $\dfrac{OQ.OA}{OE^2} = 1$ or $OQ = \dfrac{OE^2}{OA} = \dfrac{x^2}{a-x}$.

$$EQ = OE - OQ = x - \frac{x^2}{a-x} = \frac{ax - 2x^2}{a-x}.$$

By property of similar triangles $\dfrac{PQ}{BD} = \dfrac{AQ}{AD}$.

i.e., $PQ = \dfrac{BD}{AD}.AQ$. i.e., $PQ = \dfrac{b}{a}AQ$.

But $\dfrac{OQ^2}{x^2} + \dfrac{PQ^2}{y^2} = 1$,

i.e., $$\frac{PQ^2}{y^2} = 1 - \frac{OQ^2}{x^2} = \frac{x^2 - OQ^2}{x^2} = \frac{(x-OQ)(x+OQ)}{x^2}$$

$$= \frac{EQ.QD}{x^2}.$$

$$PQ^2 = \frac{y^2}{x^2}EQ.QD.$$

i.e., $\dfrac{b^2}{a^2}AQ^2 = \dfrac{b^2}{a^2}(AD-ED+EQ)^2 = \dfrac{y^2}{x^2}\dfrac{ax-2x^2}{a-x}(AD-AQ)$.

i.e., $$\frac{y^2}{x^2}\frac{ax-2x^2}{a-x}\left(a - a + 2x - \frac{ax-2x^2}{a-x}\right)$$

$$= \frac{b^2}{a^2}\left(a - 2x + \frac{ax-2x^2}{a-x}\right)^2$$

i.e., $$\frac{y^2}{x^2}x\frac{(a-2x)}{a-x}\frac{(2x(a-x)-x(a-2x))}{a-x}$$

$$= \frac{b^2}{a^2}\frac{((a-2x)(a-x)+x(a-2x))^2}{(a-x)^2}.$$

$$\text{i.e.,} \quad \frac{y^2}{x}\frac{(a-2x)}{(a-x)}\frac{ax}{a-x} = \frac{b^2}{a^2}\left(\frac{a^2(a-2x)^2}{(a-x)^2}\right)$$

$$\text{i.e.,} \quad y^2 = \frac{b^2}{a}(a-2x) \; ; \quad \text{i.e.,} \quad y = \frac{b}{\sqrt{a}}\sqrt{a-2x}\,.$$

$$\text{Area of the ellipse} = \pi yx = \pi\frac{b}{\sqrt{a}}x\sqrt{a-2x}.$$

Thus, we are to maximize $x\sqrt{a-2x}$ or, equivalently, $x^2(a-2x) = ax^2 - 2x^3 = 2(\frac{a}{2}x^2 - x^3) = 2\frac{a^3}{2^3}\left(\left(\frac{2x}{a}\right)^2 - \left(\frac{2x}{a}\right)^3\right)$ so that it suffices to maximize $\left(\frac{2x}{a}\right)^2 - \left(\frac{2x}{a}\right)^3$ or $y^2 - y^3$ where $y = \frac{2x}{a}$. By solution of Problem 2 of this chapter, this is maximum when $y = \frac{2}{3}$ or $\frac{2x}{a} = \frac{2}{3}$ or $x = \frac{a}{3}$.

Problem 11. To inscribe in a segment of a parabola bounded by a chord perpendicular to the axis another parabola with the vertex at the middle point of this chord and touching this chord and with maximum area.

Solution: Let ABC be the parabola bounded by the chord AC (Fig. 2.7).

Let PDQ be the inscribed parabola touching AC at its midpoint D. Let $BD = a$ and let the latus rectum of the given parabola be $4b$. $PQ \parallel AC$. Let PQ intersect BD at R. Let $BR = x$.

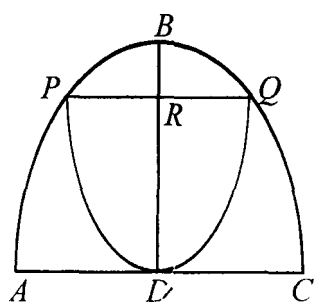


Fig. 2.7

Area of the parabola PDQ bounded by the chord $PQ = \frac{2}{3} \times x$ co-ordinate of $P \times y$ co-ordinate of P referred to DB and AC as x, y-axis respectively. By the property of the parabola $PR^2 = 4bBR$ or $PR = 2\sqrt{b.BR} = 2\sqrt{bx}$.

Area of parabola $PDQ = \frac{2}{3} \times DR \times PR = \frac{2}{3}(a-x)2\sqrt{bx} = \frac{4}{3}\sqrt{b}\sqrt{x}(a-x)$.

It suffices to maximize $\sqrt{x}(a-x) = a\sqrt{x} - x^{\frac{3}{2}} = a^{\frac{3}{2}}\left(\sqrt{\frac{x}{a}} - \left(\frac{x}{a}\right)^{\frac{3}{2}}\right)$. Putting $\sqrt{\frac{x}{a}} = y$, we are to maximize $y - y^3$. By solution of Problem 1 in this chapter, the maximum is reached when $y = \frac{1}{\sqrt{3}}$, i.e., $\sqrt{\frac{x}{a}} = \frac{1}{\sqrt{3}}$ or $x = \frac{a}{3}$.

Problem 12. To inscribe a cone of maximum volume in a given sphere.

Solution: Let ABC be the cone inscribed in the sphere of given radius a. Let D be the centre of its base. Let $BD = x$ (see fig 2.8).

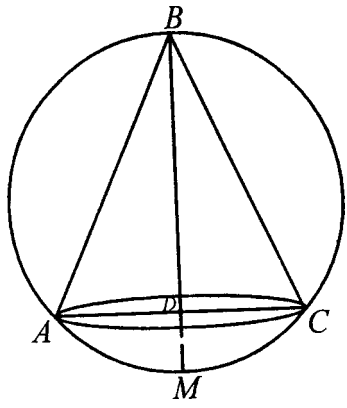


Fig. 2.8

BM is the diameter of the sphere $= 2a$. $DM = 2a - x$. By secant theorem for a circle $AD.DC = AD^2 = BD.DM = x.DM = x(2a-x)$. Volume of the cone $= \frac{1}{3}\pi x(2a-x).x = \frac{\pi}{3}(2ax^2 - x^3)$. It suffices to maximize $2ax^2 - x^3$. Write $2ax^2 - x^3 = (2a)^3\left(\left(\frac{x}{2a}\right)^2 - \left(\frac{x}{2a}\right)^3\right)$. It suffices to maximize

$y^2 - y^3$ where $y - \frac{x}{2a}$. By Problem 2, Chapter-II, the solution $y = \frac{x}{2a} = \frac{2}{3}$ or $x = \frac{4a}{3}$.

The procedure is to take a diameter of the sphere and choose a point $\frac{4}{3}$ of the radius from one of its extremities. Then take the cross section of the sphere through this point perpendicular to the diameter, construct the cone with this cross section as base and the extremity of the diameter as vertex.

Problem 13. To find the cylinder closed at both ends of maximum volume with prescribed surface area.

Solution: Let x be the radius of the base and y the height of the cylinder. Let s be the given surface area. Then s =curved surface $+2\pi x^2$; curved surface $= s - 2\pi x^2$. Height $y = \frac{s - 2\pi x^2}{2\pi x}$:

$$\text{Volume} = \pi x^2 \times \frac{s - 2\pi x^2}{2\pi x} = \frac{1}{2}x(s - 2\pi x^2).$$

We are to maximize

$$x(s - 2\pi x^2) = xs - 2\pi x^3 = 2\pi\left(\frac{s}{2\pi}x - x^3\right)$$

$$= 2\pi\left(\frac{s}{2}\right)^{\frac{3}{2}}\left[\frac{x}{\left(\frac{s}{2\pi}\right)^{\frac{1}{2}}} - \frac{x^3}{\left(\frac{s}{2\pi}\right)^{\frac{3}{2}}}\right].$$

It suffices to maximize $y - y^3$ where $y = \frac{x}{\left(\frac{s}{2\pi}\right)^{\frac{1}{2}}}$.

From Problem 1 the solution is $y = \frac{x}{\left(\frac{s}{2\pi}\right)^{\frac{1}{2}}} = \frac{1}{\sqrt{3}}$ or

$x = \frac{\sqrt{s}}{\sqrt{6\pi}}$.

Problem 14. To show that the height of the cylinder of maximum volume inscribed in a given sphere of radius r is $\frac{2r}{\sqrt{3}}$.

Solution: The axis MN of the cylinder is along a diameter of the sphere and the centre O of the sphere is the midpoint of MN. (See Fig. 2.9).

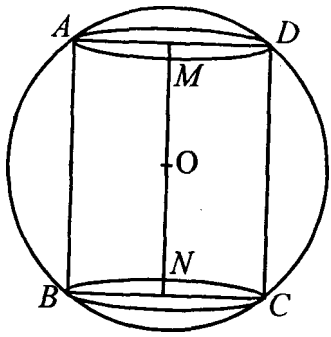


Fig. 2.9

Let $ABCD$ be the cross section of the sphere and cylinder by a plane through the axis of the cylinder. Let $MN = 2x$ so that $ON = x$, $BN = \sqrt{r^2 - x^2}$ if r is the radius of the sphere. BN is the radius of the base of the cylinder. The volume of the cylinder $= \pi(r^2 - x^2).2x = 2\pi(r^2x - x^3)$.

By Solution to Problem 1, Chapter-II, since $r^2x - x^3 = r^3\left(\frac{x}{r} - \left(\frac{x}{r}\right)^3\right)$ the maximum value is given by $\frac{x}{r} = \frac{1}{\sqrt{3}}$ or $x = \frac{r}{\sqrt{3}}$ so that the height $2x = \frac{2r}{\sqrt{3}}$.

Problem 15. To find the height of a flame above a horizontal plane which will illuminate a small object at a given distance from its projection on the plane to the maximum.

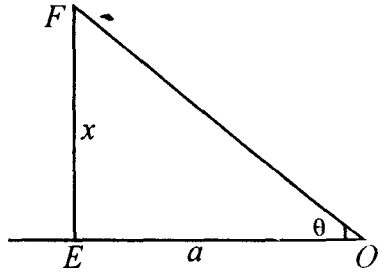


Fig. 2.10

Solution: Let F be the flame considered as a point above the horizontal plane at a height x from it. Let O be the illuminated object,

E the foot of the perpendicular from F to the plane. (See Fig. 2.10). Given $EO = a$, constant, let $FE = x$. From rules of optics the intensity of illumination of O is proportional to

$$\frac{1}{FO^2} \times \sin F\hat{O}E = \left[\frac{1}{\left(\sqrt{a^2+x^2}\right)^2}\right] \frac{x}{\sqrt{a^2+x^2}}$$

i.e., to $\frac{1}{\sqrt{a^2+x^2}} \frac{x}{a^2+x^2}$, i.e., to $\sin\theta . \frac{\cos^2\theta}{a^2}$, i.e., to $\sin\theta . \cos^2\theta$, i.e., to $y(1-y^2)$ if $y = \sin\theta$. We are to maximize $y - y^3$. By Problem 1, Chapter-II, this is reached when $y = \frac{1}{\sqrt{3}}$. i.e., $\sin\theta = \frac{1}{\sqrt{3}}$. Thus height of the flame is $x = a\tan\theta = a\sqrt{\sec^2\theta - 1} = a\sqrt{\frac{3}{2} - 1} = \frac{a}{\sqrt{2}}$.

Problem 16. To partition 12 so that the product of the lesser summand with the square of the larger one is maximum.

Solution : To find x such that $x^2(12-x)$ is maximum, given that $x > 12 - x$ or $x > 6$. i.e., to maximize $12x^2 - x^3 = 12^3\left(\left(\frac{x}{12}\right)^2 - \left(\frac{x}{12}\right)^3\right),$. i.e., to maximize $\left(\frac{x}{12}\right)^2 - \left(\frac{x}{12}\right)^3$. By Solution to Problem 2, Chapter-II, the maximum is reached when $\frac{x}{12} = \frac{2}{3}$ or $x = 8$.

Problem 17. To find the values of x for which $r = \frac{x^3}{3} - \frac{3x^2}{2} + 2x$ is maximum or minimum.

Solution: Multiplying the expression by 6 we are to find

the values of x for which $2x^3 - 9x^2 + 12x$ is extreme. If we take all real values there is no maximum or minimum since the values of the expression lie in $(-\infty, +\infty)$. The polynomial can be written as $2(x-1)^3 - 3(x-1)^2 + 5$ so that we are to find extremum of $2(x-1)^3 - 3(x-1)^2$ $= 2\left[(x-1)^3 - \frac{3}{2}(x-1)^2\right] = 2 \times \frac{3^3}{2^3}\left[\left(\frac{x-1}{3/2}\right)^3 - \left(\frac{x-1}{3/2}\right)^2\right]$ or equivalently of $y^3 - y^2$, where $y = \frac{x-1}{3/2}$. By Problem 2, Chapter-II, the maximum of $y^2 - y^3$ is reached in $0 \le y \le 1$ when $y = \frac{2}{3}$, i.e., when $x - 1 = \frac{3}{2}\cdot\frac{2}{3} = 1$ i.e., when $x = 2$. So r is minimum when $x = 2$.

The minimum value of $y^2 - y^3$ in $[0, 1]$ is clearly 0 and is reached when $y = 0$ or 1, i.e., when $x - 1 = 0$, i.e., $x = 1$, $\frac{x-1}{3/2} = 1$ or $x - 1 = 3/2$ or $x = \frac{5}{2}$. So r is maximum when $x = 1$ or $x = \frac{5}{2}$.

Note: The given polynomial vanishes at $x = 0$, takes the value $\frac{5}{6}$ at $x = 1$ and at $x = \frac{5}{2}$ and takes the minimum value of $\frac{2}{3}$ at $x = 2$. (See Fig. 2.11).

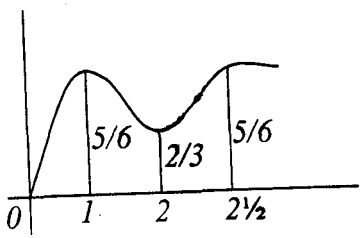


Fig. 2.11

Problem 18. To find the number such that its square and twenty one times itself subtracted from its cube is maximum.

Solution: To maximize $x^3 - x^2 - 21x = 8(x-3)^2 + (x-3)^3 + 45 = 8(3-x)^2 - (3-x)^3 + 45$ or equivalently $8(3-x)^2 -$

$(3-x)^3 = 8^3\left[\frac{(3-x)^2}{8^2} - \frac{(3-x)^3}{8^3}\right]$ or equivalently $y^2 - y^3$ where $y = \frac{3-x}{8}$. By solution to Problem 2, Chapter-II, the maximum is reached when $\frac{3-x}{8} = \frac{2}{3}$, i.e., $9 - 3x = 16$, i.e., $3x = -7$, i.e., $x = -\frac{7}{3}$.

Note: The graph of the function $f(x) = x^3 - x^2 - 21x$ is as in Fig. 2.12. It attains the value 0 at $x = 0$ and at $\frac{1 \pm \sqrt{1+84}}{2} = \frac{1}{2} \pm \frac{\sqrt{85}}{2}$ (roughly near 5 and -4), it is positive in $(-4, 0)$ and negative in $(0, 5)$ increasing to ∞ in $(5, \infty)$ and decreasing to $-\infty$ in $(-\infty, -4)$. Thus the maximum value is reached at $x = -\frac{7}{3}$ in $(-4, 0)$ and the minimum values is reached in $(0, 5)$. There is an error in the author's solution. The value at $x = 3$ is $27 - 9 - 63 = -45$ is not the maximum value in $(-4, 5)$ but the minimum value.

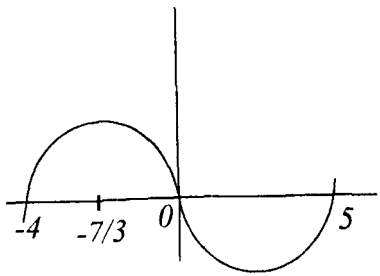


Fig. 2.12

Problem 19. To find the ellipse of maximum area which is obtained as a cross section of a given right circular cone.

Solution: Let ABD be a vertical cross-section of the right circular cone (Fig. 2.13) with C as the centre of its base.

If a plane intersects the cone and is not parallel to one of its generators the section is an ellipse. The cross section obtained by a plane parallel to this plane passing through B is a larger ellipse. Thus let us assume that the ellipse considered has B as one of the ends of its major axis. Let the other end of this axis be

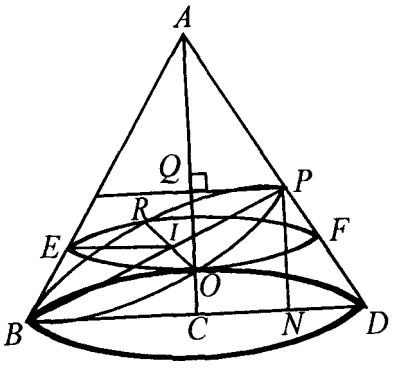


Fig. 2.13

P on the cone. Let I be the centre of the ellipse and RO its minor axis. Let E, F be the points on the vertical section on the cone on the circle got by the cross section by the plane parallel to the base through RO. Let $AC = a$, $BC = b$, constants, and let PQ, PN be perpendicular to AC, BD respectively, Q lying on AC and N on BD.

Let $PN = y$, $RO = 2n$, $BP = 2m$, $CN = x$, all variables. By similar triangles,

$$\frac{PB}{IB} = \frac{PQ}{EI}, \quad \text{i.e.,} \quad \frac{PQ}{EI} = \frac{2}{1}. \quad \frac{BP}{IP} = \frac{2}{1} = \frac{BD}{IF}$$

i.e., $BP = 2(IP)$, $BD = 2(IF)$. So, $PQ \times BD = 2EI \times 2IF = 4IE \times IF = 4n^2$. Hence $2n = \sqrt{PQ \times BD} = \sqrt{2x \times 2b} = 2\sqrt{bx}$. Also $2m = \sqrt{BN^2 + PN^2} = \sqrt{(b+x)^2 + PN^2}$.

But $PN = AC \times \frac{DN}{CD} = \frac{a(b-x)}{b}$, i.e., $2m = \sqrt{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2}$. Area of the ellipse is πmn $\frac{\pi\sqrt{bx}}{2}\sqrt{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2}$. We are therefore to maximize

(after squaring and leaving constant multipliers)

$$x\left\{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2\right\} = \frac{a^2+b^2}{b^2}x^3 - \frac{2(a^2-b^2)}{b}x^2 + (a^2+b^2)x.$$

Equivalently, we are to maximize

$$x^3 - \frac{2b(a^2-b^2)}{a^2+b^2}x^2 + b^2x.$$

Writing

$$\frac{2b(a^2-b^2)}{a^2+b^2} = q,$$

we are to maximize

$$x^3 - qx^2 + b^2x = x(x^2 - qx + b^2)$$

and the quadratic has no real roots since

$$q^2 - 4b^2 = \frac{4b^2(a^2-b^2)^2}{(a^2+b^2)^2} - 4b^2 = 4b^2\left[\frac{(a^2-b^2)^2 - (a^2+b^2)^2}{(a^2+b^2)^2}\right]$$

$$= \frac{4b^2}{(a^2+b^2)^2}[-4a^2b^2] < 0.$$

Consider $x^3 - qx^2 + b^2x = r$, $r \geq 0$ (Note that $x^3 - qx^2 + b^2x < 0$ for $x < 0$) for $x \geq 0$. $x^3 - qx^2 + b^2x - r = 0$ is a cubic whose real root is positive for $r > 0$. If v is this root, then $x^3 - qx^2 + b^2x - r = (x-v)[x^2 + (v-q)x + v^2 - vq + b^2]$ where the remainder, when divided by $x - v$, $v(v^2 - vq + b^2) - r = 0$

or $r = v(v^2 - vq + b^2)$ (Note that the remainder is got using the remainder theorem). From the quotient we have $x^2 + (v-q)x = -\frac{r}{v}$ the value of x being the one corresponding to the maximum of the expression

$$\left(x + \frac{v-q}{2}\right)^2 - \left(\frac{v-q}{2}\right)^2 = -\frac{r}{v}$$

or

$$\frac{r}{v} = \left(\frac{v-q}{2}\right)^2 - \left(x - \frac{v-q}{2}\right)^2$$

so that r is maximum when $x = \frac{v-q}{2}$. Then $\frac{r}{v} = \left(\frac{v-q}{2}\right)^2$ or $4r = v(v-q)^2 = v(v^2 - 2qv + q^2) = 4v(v^2 - vq + b^2)$ so that, since $v > 0$, $v^2 - 2qv + q^2 = 4(v^2 - vq + b^2)$ i.e., $3v^2 - 2qv + 4b^2 - q^2 = 0$. v has a real value when $4q^2 - 4\times 3(4b^2 - q^2) \geq 0$, i.e., $16q^2 - 48b^2 \geq 0$, i.e., $q^2 - 3b^2 \geq 0$. i.e., $4b^2\frac{(a^2-b^2)^2}{(a^2+b^2)^2} - 3b^2 \geq 0$, i.e., $4(a^2-b^2)^2 - 3(a^2+b^2)^2 \geq 0$, i.e., $a^4 - 14a^2b^2 + b^4 \geq 0$, i.e., $a^4 - 14a^2b^2 + 49b^4 \geq 48b^4$. The minimum requirement is when equality holds, i.e., $(a^2 - 7b^2)^2 = 48b^4$ or when $a^2 - 7b^2 = \pm 4\sqrt{3}b^2$ or $a^2 = (7 \pm 4\sqrt{3})b^2$. Then

$$x = -(\frac{q}{3} - q)/2 = \frac{q}{3} = \frac{2b(a^2-b^2)}{3(a^2+b^2)} = \frac{2b}{3}\frac{(6 \pm 4\sqrt{3})}{(8 \pm 4\sqrt{3})} = \frac{b}{3}\frac{(3 \pm 2\sqrt{3})}{(2 \pm \sqrt{3})}.$$

Problem 20. The corner of a page of a book is turned back so as to reach other edge, find when the length of the crease made is minimum.

Solution: Let $MNAB$ be the page. (See. fig. 2.14). Let PQ be the crease when the page is folded so that A goes to A' on MB. Then $P\hat{A}'Q = 90°$. $QA'PA$ is a cyclic quadrilateral. Let $PA = x$ (variable), $AB = a$ (constant). $A'Q = AQ$, $AP = A'P$. By Pythagoras theorem:

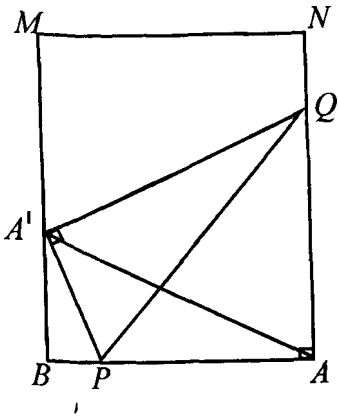


Fig. 2.14

$$AA'^2 = A'B^2 + AB^2 = A'B^2 + (AP + BP)^2 = A'B^2 + BP^2 + AP^2 + 2AP.BP = A'P^2 + AP^2 + 2AP.BP = 2x^2 + 2x(a - x) = 2ax\,.$$

So $A'A = \sqrt{2ax}$. $AQ^2 = QP^2 - AP^2$ and by Ptolemy's theorem $AA' \times PQ = 2AP \times AQ$ (since $AQ = A'Q$, $AP = A'P$). So, $AA'^2 \times PQ^2 = 4AP^2 \times AQ^2 = 4AP^2(QP^2 - AP^2)$ $= 4AP^2 \times QP^2 - 4AP^4$ or $4AP^4 = (4AP^2 - AA'^2) \times QP^2$.

i.e., $4x^4 = (4x^2 - 2ax)PQ$ or $PQ = \dfrac{4x^4}{2x(2x-a)} = \dfrac{2x^3}{2x-a}$ is to be minimized. Equivalently, $\dfrac{2x-a}{2x^3}$ is to be maximized, i.e., $\dfrac{1}{x^2} - \dfrac{a}{2x^3}$ is to be maximized. i.e., $\dfrac{a}{2}\left[\dfrac{2}{ax^2} - \dfrac{1}{x^3}\right]$ or equivalently

$\dfrac{2}{ax^2} - \dfrac{1}{x^3}$ is to be maximized, i.e., $\dfrac{2^3}{a^3}\left[\left(\dfrac{a}{2x}\right)^2 - \left(\dfrac{a}{2x}\right)^3\right]$ is to be maximized or $\left(\dfrac{a}{2x}\right)^2 - \left(\dfrac{a}{2x}\right)^3 = y^2 - y^3$ where $y = \dfrac{a}{2x}$

is to be maximized. By Problem 2, Chapter-II, the solution is $y = \frac{a}{2x} = \frac{2}{3}$ or $x = \frac{3a}{4}$.

Problem 21. To find the position of the planet Venus in relation to the earth when her light is the greatest.

Solution: We take sun S and earth E as fixed points (See. Fig. 2.15) and Venus as a circular cross section. Noting that Venus moves in an elliptic orbit around S, we observe that if it is in a position beyond E and S, its full cross section which is illuminated by the sun is seen as a circular disc and is not the brightest since it is near the sun and away from earth. If it is in its nearest position to the sun between the earth and the sun its illuminated part faces the sun so that it is not the brightest, then. Thus the planet should be in some position as in the figure 2.15, if it were to be the brightest. ABD is the illuminated hemisphere turned towards the sun. CBD is the hemisphere seen from the earth when AB and CD are perpendicular to SV and EV respectively. As seen from E the position of V which is visible is the hemisphere facing E cut by a plane perpendicular to EV (V being the centre of Venus). The illuminated part of the planet is the hemisphere obtained by a plane perpendicular to SV facing S.

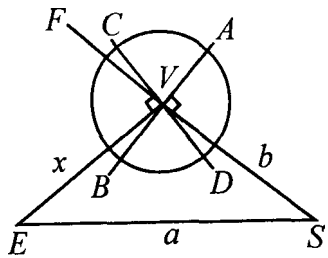


Fig. 2.15

If in the two dimensional picture in Fig 2.15 the two planes are denoted by the lines AVB, CVD respectively, the illuminated part which is visible from E corresponds to the sector BVD projected as a crescent on the plane represented by CD. The breadth of the crescent is $r(1 - \cos B\hat{V}D) = r(1 - \cos(180° - E\hat{V}S)) = r(1 + \cos E\hat{V}S)$. The illuminated part is thus proportional to $1 + \cos E\hat{V}S$. The illumination in relation to the position of earth and sun is proportional to $\frac{1 + \cos E\hat{V}S}{EV^2}$. Let the distance of earth from sun $ES = a$ and that of Venus from sun $SV = b$ so that a, b are constants. Let $VE = x$, variable. By cosine formula for ΔEVS,

$$\cos E\hat{V}S = \frac{x^2 + b^2 - a^2}{2bx}.$$

$$\frac{1 + \cos E\hat{V}S}{EV^2} = \frac{1 + \frac{x^2 + b^2 - a^2}{2bx}}{x^2} = \frac{2bx + x^2 + b^2 - a^2}{2bx^3}.$$

We are to maximize $\frac{2bx + x^2 + b^2 - a^2}{x^3} = r$, say.

Then $rx^3 - x^2 - 2bx + a^2 - b^2 = 0$. Putting $y = \frac{1}{x}$, $(a^2 - b^2)y^3 - 2by^2 - y + r = 0$.

i.e., $$y^3 - \frac{2b}{a^2 - b^2}y^2 - \frac{y}{a^2 - b^2} + \frac{r}{a^2 - b^2} = 0.$$

Put $\frac{2b}{a^2 - b^2} = m$, $\frac{1}{a^2 - b^2} = n$ and write $\frac{r}{a^2 - b^2} = v$. Then we have $y^3 - my^2 - ny + v = 0$ (v being to be maximized). If c is the negative real root of this equation (assuming that v has a positive maximum), then $y^3 - my^2 - ny + v = (y + c)\{y^2 - (c + m)y + c^2 + mc - n\}$ and $v = c(c^2 + cm - n)$ or $\frac{v}{c} = c^2 + cm - n$. So, $y^2 - (c + m)y = -\frac{v}{c}$,

i.e., $\left(y - \frac{c+m}{2}\right)^2 = -\frac{v}{c} + \frac{(c+m)^2}{4}$ i.e., $\frac{v}{c} = \frac{(c+m)^2}{4} - \left(y - \frac{c+m}{2}\right)^2$. Hence v is maximum when $y = \frac{c+m}{2}$.

Then $v = c\frac{(c+m)^2}{4}$, so that $\left(\frac{c+m}{2}\right)^2 = c^2 + cm - n$, i.e., $c^2 + 2mc + m^2 = 4c^2 + 4cm - 4n$, i.e., $3c^2 + 2cm - m^2 - 4n = 0$. If c should have a real value $4m^2 + 4 \times 3(m^2 + 4n) \geq 0$, which is true when $a > b$. Then

$$c = \frac{-2m \pm \sqrt{16m^2 + 48n}}{6} = \frac{-m \pm 2\sqrt{m^2 + 3n}}{3}.$$

and

$$y = \frac{c+m}{2} = \frac{1}{2}\left(\frac{-m + 2\sqrt{m^2 + 3n}}{3} + m\right) = \frac{m + \sqrt{m^2 + 3n}}{3}$$

(ignoring the negative value for y).

$$\begin{aligned} &= \frac{1}{3}\left[\frac{2b}{a^2 - b^2} + \frac{1}{a^2 - b^2} + \sqrt{3a^2 + b^2}\right] \\ &= \frac{1}{3(a^2 - b^2)}\left[2b + \sqrt{3a^2 + b^2}\right] \\ &= \frac{2b + \sqrt{3a^2 + b^2}}{3a^2 + b^2 - 4b^2} \\ &= \frac{1}{\sqrt{3a^2 + b^2} - 2b} \end{aligned}$$

So $x = \frac{1}{y} = \sqrt{3a^2 + b^2} - 2b (\geq 0$ under the assumption a $\geq b)$.

Problem 22. To find the diameter of a water wheel so as to receive the maximum effect from a stream of water of 12 feet fall.

Solution: As in Fig. 2.16, let C be the centre of the wheel CD, CB the horizontal and vertical diameters respectively. Let A be on BC so that it is at a height $12'$ from B (the height of the fall). Let the stream meet the the wheel at D first and fall vertically thereafter. Write $AB = a(= 12)$.

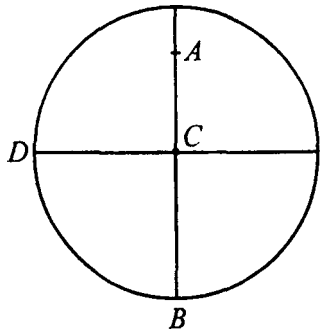


Fig. 2.16

Let $CD = CB = x$. The only force acting on the fall is gravity g so that acceleration is constant. (using "$F = ma$"). Thus velocity is proportional to time and distance travelled is proportional to square of time and so to square of velocity. Thus velocity is proportional to the square root of the distance travelled upto D, i.e., $\sqrt{a-x}$. The effect is known to be proportional to velocity and the radius of the wheel (the length of the lever). i.e., to $x\sqrt{a-x}$ which we are to maximize. Thus we are to maximize $ax^2 - x^3$, i.e., to maximize $\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^3$. By Problem 2, Chapter-II, the maximum is reached when $\frac{x}{a} = \frac{2}{3}$ or $x = \frac{2a}{3} = \frac{2 \times 12}{3} = 8$ feet.

Remarks 1: If the flow of water is such that it gets in touch with the wheel at its bottom (See Fig. 2.17) the effect is known to increase with the radius of the wheel.

2. If the flow meets the wheel at the top, then we have to maximize $x\sqrt{a-2x}$ and the maximum is reached when $x = 4$.

3. Resistance and the volume of the buckets in the wheel have been ignored.

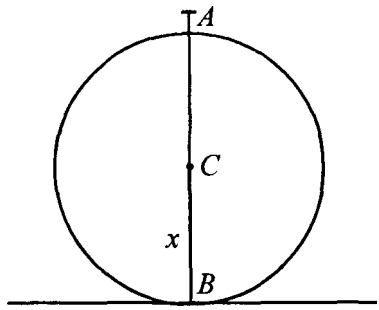


Fig. 2.17

Problem 23. To determine the strongest angle of position of a pair of gates for the lock on a canal or river.

Solution: Let AC, BC be the two gates, $AC = BC$ meeting at C projecting out against the pressure of water, AB being the breadth of the canal or river. (Fig. 2.18). The pressure on the gate AC is proportional to the area of the gate, and the height being fixed to the depth of the flow, to the length AC and the length of the lever operating at A, viz., AC again i.e., the pressure on the gate is as AC^2. Extend AC and draw BD perpendicular to AC to meet it at D. The pressure on the gate BC is proportional to BC and can be resolved in the perpendicular directions AC and BD. By triangle of forces these components are proportional to CD and BD

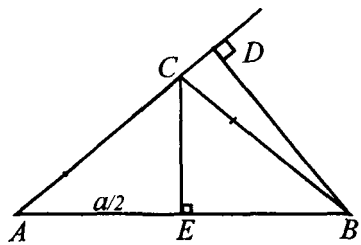


Fig. 2.18

respectively. Now ΔACE and ΔABD are similar where E is the midpoint of AB so that $CE \perp AB$. Hence $\frac{AC}{AE} = \frac{AB}{AD}$. AB, AE being constants AC is inversely proportional to AD. So AC^2 is inversely proportional to AD^2. The resistance on AC is increased by the resistance on BC resolved perpendicular to AC, viz., BD. Thus the resistance on both gates is proportional to $AD^2 \times BD$. But $AD^2 = AB^2 - BD^2$ so that the resistance is proportional to $x(a^2 - x^2)$ where $a = AB$ (constant) and $BD = x$ (variable). We are to maximize $a^2x - x^3 = a^3\left[\frac{x}{a} - \left(\frac{x}{a}\right)^3\right]$. From Problem 1, Chapter-II, the maximum is reached when $\frac{x}{a} = \frac{1}{\sqrt{3}}$ or $x = \frac{a}{\sqrt{3}}$. i.e., $\sin A = \frac{a}{\sqrt{3}}\frac{1}{a} = \frac{1}{\sqrt{3}}$. $A = \sin^{-1}\frac{1}{\sqrt{3}} = B = 35°16'$, $A\hat{C}B = 109°28'$.

Problem 24. To find the edge of a cube which would displace the maximum water when dropped vertically in a conical vessel held vertically by its vertex full of water, the depth of the water being a and the diameter of the mouth of the vessel being $2b$.

Solution: Clearly maximum displacement can be obtained only by inserting the cube in such a way that the line joining the centers of its immersed face and the opposite face coincides with the attitude of the cone and its four edges lie on the slant surface of the cone.

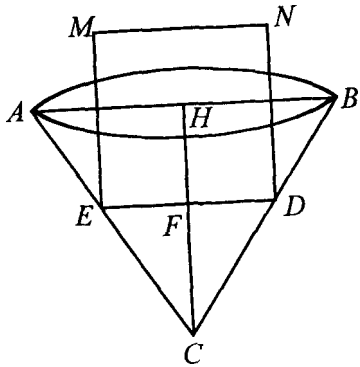


Fig. 2.19

Let ACB be the cross section of the cone and $MEDN$ that of the cube C being the vertex of the cone. Let F be the centre of the immersed face and H the centre of the base of the cone. (See Fig. 2.19). Let $FC = x$ (variable). $\Delta AHC \,|||\, \Delta EFC$ so that $\frac{HC}{AH} = \frac{FC}{EF}$ or $EF = \frac{bx}{a}$. Hence $ED = 2EF = \frac{2bx}{a}$. Thus the area of the base of the cube is $\frac{4b^2x^2}{a^2}$. The water displaced has volume equal to $\frac{4b^2x^2}{a^2} \times FH = \frac{4b^2x^2}{a^2}(HC - FC) = \frac{4b^2x^2}{a^2}(a-x)$. Thus the problem reduces to maximizing $x^2(a-x) = ax^2 - x^3 = a^3\left(\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^3\right)$. By Problem 2, Chapter II, this is obtained when $\frac{x}{a} = \frac{2}{3}$ or $x = \frac{2a}{3}$. Thus the side of the cube is $ED = \frac{2bx}{a} = \frac{2b}{a} \times \frac{2a}{3} = \frac{4b}{3}$.

Remark: There is an error in the author's value of ED.

Problem 25. To find the size of a spherical ball which will displace maximum amount of water when dropped in a conical vessel full of water held vertically with vertex down to get in touch with its slant surface.

Solution: Let ACB be the vertical section of the cone, C being its vertex. (Fig. 2.20), G the centre of the base. Let F be the centre of the sphere lying on CG. Let D, E be the points of contact of the sphere with the slant surface of the cone in the vertical section.

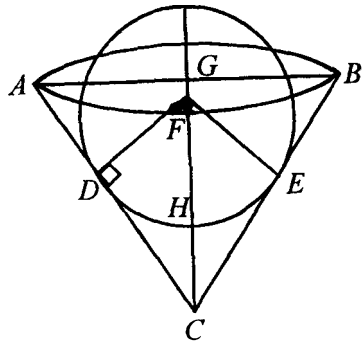


Fig. 2.20

Let $AG = GB = a$, $GC = b$, constants $AC = \sqrt{AG^2 + CG^2} = \sqrt{a^2 + b^2} = c$, say. $DF = FH = FE = x$ (variable) = radius of the ball. ΔACG and ΔFCD are similar so that $\frac{AG}{AC} = \frac{DF}{FC}$, i.e., $\frac{a}{c} = \frac{x}{FC}$ or $FC = \frac{cx}{a}$. So, $GF = GC - FC = b - \frac{cx}{a}$ and $GH = GF + FH = b - \frac{cx}{a} + x =$ height of the segment of the sphere immersed in water. Volume of the immersed part is therefore

$$
\begin{aligned}
&(6DF - 2GH) \times GH^2 \times \frac{\pi}{6} \\
&\quad = \frac{\pi}{6}\left[6x - 2b + \frac{2cx}{a} - 2x\right] \times \left(b + x - \frac{cx}{a}\right)^2 \\
&\quad = \frac{\pi}{6}\left[\left(\frac{2c}{a} + 4\right)x - 2b\right]\left[\left(1 - \frac{c}{a}\right)x + b\right]^2 \\
&\quad = \frac{\pi}{3}\left[\left(2 + \frac{c}{a}\right)x - b\right]\left(b - \frac{c-a}{a}x\right)^2
\end{aligned}
$$

noting that $c > a$. Put $b - \frac{c-a}{a}x = y$. Then $x = \frac{(b-y)a}{c-a}$ and

$$
\begin{aligned}
\frac{2a+c}{a}x - b &= \frac{(2a+c)}{a}\frac{(b-y)a}{(c-a)} - b \\
&= \frac{3a^2b - a(2a+c)y}{a(c-a)} = \frac{3ab - (2a+c)y}{(c-a)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\frac{2a+c}{a}x - b\right)\left(b - \frac{c-a}{a}x\right)^2 &= \frac{3aby^2 - (2a+c)y^3}{c-a} \\
&= \frac{2a+c}{c-a}\left[\frac{3ab}{2a+c}y^2 - y^3\right]
\end{aligned}
$$

is to be maximized, i.e., $\frac{3ab}{2a+c}y^2 - y^3 = Ay^2 - y^3$ where $A = \frac{3ab}{2a+c}$, is to be maximized. i.e.,

$$A^3\left[\left(\frac{y}{A}\right)^2 - \left(\frac{y}{A}\right)^3\right]$$

is to be maximized. From Problem 2, Chapter II, this is done when $\frac{y}{A} = \frac{2}{3}$ or $y = \frac{2A}{3}$ i.e., $y = \frac{2ab}{2a+c}$. So

$$x = \frac{(b-y)}{c-a}a = \frac{(b - \frac{2ab}{2a+c})a}{c-a} = \frac{abc}{(c-a)(2a+c)}$$

is the radius of the desired sphere.

Problem 26. To find x for which $\frac{(x-1)^2}{(x+1)^3}$ is maximum.

Solution: Let $x+1 = \frac{1}{y}$. Then $(x-1) = \frac{1}{y} - 2 = \frac{1-2y}{y}$. Hence

$$\frac{(x-1)^2}{(x+1)^3} = \frac{(1-2y)^2}{y^2} \times y^3 = y(1-2y)^2 = Y^2\frac{(1-Y)}{2}$$

where $Y = 1-2y$. Thus we are to maximize $Y^2 - Y^3$ which gives the solution $Y = \frac{2}{3}$, i.e., $1-2y = \frac{2}{3}$ or $2y = \frac{1}{3}$, $y = \frac{1}{6}$ i.e., $x+1 = \frac{1}{y} = 6$ or $x = 5$.

Problem 27. To find the dimension of a rectangular beam to be cut from the trunk of a tree in cylindrical shape so that used as a beam it has the greatest power of suspension.

Solution: Let $ACBM$ be the circular outer section of the trunk. Let the rectangle $ACBM$ represent the cross section of the beam. Let AB be the diameter of the circle. Let R

and N be the feet of the perpendiculars from M, C respectively to AB See Fig.2.21. So $AB = a$ (constant), $AC = x =$ breadth of the beam, a variable. Depth of beam $= BC = \sqrt{a^2 - x^2}$. Power of suspension $= \dfrac{f \times \text{breadth} \times \text{depth}^2}{\text{length}}$, f being the strength of the wood, a constant. Thus the power is proportional to $x\left(\sqrt{a^2 - x^2}\right)^2 = a^2x - x^3$ which is maximum when $\dfrac{x}{a} = \dfrac{1}{\sqrt{3}}$, by Problem 1, Chapter II. i.e., $x = \dfrac{a}{\sqrt{3}}$.

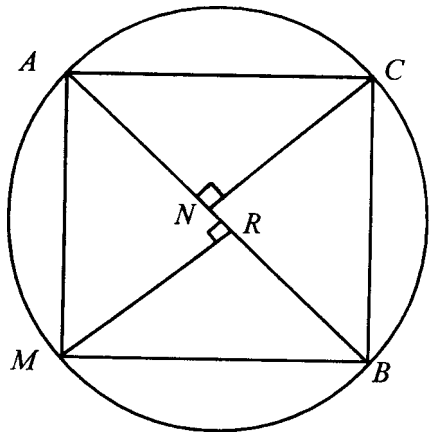


Fig. 2.21

By similar triangles $\dfrac{AB}{AC} = \dfrac{AC}{AN}$; i.e., $\dfrac{a}{x} = \dfrac{x}{AN}$.

i.e., $AN = \dfrac{x^2}{a} = \dfrac{a^2}{3a} = \dfrac{a}{3}$. Also $\dfrac{AB}{BM} = \dfrac{BM}{BR}$, i.e., $\dfrac{a}{x} = \dfrac{x}{BR}$, i.e., $BR = \dfrac{x^2}{a} = \dfrac{a}{3}$ and so $NR = AB - RB - AN = a - \dfrac{a}{3} - \dfrac{a}{3} = \dfrac{a}{3}$.

Thus the construction of the cross section of the beam from the trunk is as follows: Divide the diameter of the trunk into three equal parts and from the two points of division erect the perpendiculars to the diameter. The points at which the two perpendiculars meet the periphery of the trunk and the two ends of the diameter decide the cross section of the beam.

CHAPTER III

Problems of Maxima, Minima in the Solutions of which Equations of Fourth to Seventh Degree are used

Problem 1. Find $x, 0 < x < 1$ such that $x^3 - x^4$ is maximum.

Solution: Write $x^3 - x^4 = r$ for the maximum value r. Then $x^4 - x^3 + r = 0$. Now $x^4 - x^3$ takes nonpositive values in $[0, 1]$ and is 0 at 0 and 1. Thus, when $-r$ is its minimum value the graph of $y = x^4 - x^3 + r$ cuts the x-axis at least at one point, i.e., $x^4 - x^3 + r = 0$ has one real root and hence another real root too since it cannot have three nonreal roots: If c and d are these roots and $(x - c)(x - d) = x^2 - ax + b$, then

$$x^4 - x^3 + r = (x^2 - ax + b)\left\{x^2 + (a-1)x + a^2 - a - b\right\}$$

$$-b(a-1)x + a(a^2 - a - b)x + r - b(a^2 - a - b). \qquad (1)$$

by long division, but since $x^2 - ax + b$ is a factor of $x^4 - x^3 + r$, we should have

$$\left.\begin{array}{lll} \text{(i)} & -b(a-1) + a(a^2 - a - b) & = 0 \\ \text{(ii)} & r - b(a^2 - a - b) & = 0 \end{array}\right\} \qquad (2)$$

From (2) (i) $2ab - b = a^3 - a^2$ or $b = \dfrac{a^3 - a^2}{2a - 1}$. Hence

$$a^2 - a - b = a^2 - a - \frac{a^3 - a^2}{2a - 1}$$

$$= \frac{(2a-1)(a^2 - a) - (a^3 - a^2)}{2a - 1}$$

$$= \frac{(a^2-a)(2a-1-a)}{2a-1}$$

$$= \frac{a(a-1)(a-1)}{2a-1}$$

$$= \frac{a(a-1)^2}{2a-1}. \qquad (3)$$

Thus from (2)(ii)

$$r = b(a^2-a-b)$$

$$= \frac{a^3-a^2}{2a-1}\left(\frac{a(a-1)^2}{2a-1}\right)$$

$$= \frac{a^3(a-1)^3}{(2a-1)^2} \qquad (4)$$

From (1), (3) and (4), $x^2+(a-1)x+a^2-a-b=0$ for that value of x for which $x^4-x^3=-r$ since $x^2-ax+b\neq 0$ except for $x=c,d$; or $x^2+(a-1)x=-\frac{r}{b}$ from (2)(ii),

i.e., $$x^2+(a-1)x+\frac{(a-1)^2}{4}=\frac{(a-1)^2}{4}-\frac{r}{b}.$$

i.e., $$-\left(x-\frac{1-a}{2}\right)^2+\frac{(a-1)^2}{4}=\frac{r}{b}.$$

Thus, r is maximum (noting that $1>a,b>0$) when $x=\frac{1-a}{2}$ and then

$$r=\frac{b}{4}(a-1)^2=\frac{a^2(a-1)(a-1)^2}{4(2a-1)}=\frac{a^2(a-1)^3}{4(2a-1)}.$$

So, $\frac{a^2(a-1)^3}{4(2a-1)}=\frac{a^3(a-1)^3}{(2a-1)^2}$ by (4).

So, $4a=2a-1$ or $a=-\frac{1}{2}$ so that $x=\frac{1+\frac{1}{2}}{2}=\frac{3}{4}$.

Problem 2. To find $x, 0 < x < 1$, such that $x - x^4$ is maximum.

Solution: If r is the maximum value, since $x^4 - x$ has minimum value $-r$, vanishes at 0 and 1 and is nonpositive in $[0, 1]$, $x^4 - x + r$ is such that the graph of $y = x^4 - x + r$ cuts (or touches) the x-axis at a point. i.e., $x^4 - x + r = 0$ has two real roots, say c, d. Let $(x - c)(x - d) = x^2 - ax + b$. Then, by long division

$$x^4 - x + r = (x^2 - ax + b)\left\{x^2 + ax + a^2 - b\right\}$$

$$-(ab + 1)x + a(a^2 - b)x + r - b(a^2 - b) \tag{1}$$

Now, $(x^2 - ax + b)$ being a factor of $x^4 - x + r$, we have

$$-(ab + 1) + a(a^2 - b) = 0 \tag{2}$$

and

$$r - b(a^2 - b) = 0. \tag{3}$$

From (2),

$$2ab = a^3 - 1 \text{ or } b = \frac{a^3 - 1}{2a} \tag{4}$$

From (3)

$$\begin{aligned} r = b(a^2 - b) &= \frac{a^3 - 1}{2a}\left\{a^2 - \frac{a^3 - 1}{2a}\right\} \\ &= \frac{(a^3 - 1)}{4a^2}(2a^3 - a^3 + 1) \\ &= \frac{(a^3 - 1)(a^3 + 1)}{4a^2} \end{aligned} \tag{5}$$

Now, From (1) and (3), $x^2 + ax + a^2 - b = x^2 + ax + \frac{r}{b} = 0$ i.e., $\frac{r}{b} = \frac{a^2}{4} - (x + \frac{a}{2})^2$. It follows that (since $1 > a, b > 0$),

that r is maximum when $x = -\frac{a}{2}$. Also maximum value $r = \frac{a^2 b}{4}$. From (5) and (4)

$$\frac{(a^3-1)(a^3+1)}{4a^2} = \frac{a^2 b}{4} = \frac{a^2}{4} \times \frac{a^3-1}{2a}$$

So $2(a^3+1) = a^3$ or $a^3 = -2$ or $a = -\sqrt[3]{2}$.
Thus $x = -\frac{a}{2} = \frac{\sqrt[3]{2}}{2} = \frac{1}{\sqrt[3]{4}}$.

Problem 3. Given a parabolic arc bound by its semilatus rectum and axis, to find a tangent to the arc which forms a triangle of minimum area along with these boundaries.

Solution: Let AB be the parabolic arc, A its vertex, C the focus of the parabola, AC, the axis and BC the semilatus rectum. If ST is the desired tangent touching the arc at P, let N, M be the feet of the perpendiculars from P on the axis and latus rectum respectively (Fig. 3.1).

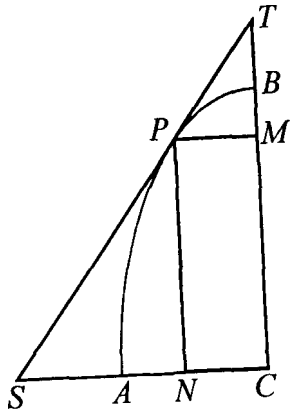


Fig. 3.1

Let $AC = a$, so that parabola has the equation $y^2 = 4ax$ referred to its axis and the tangent at the vertex A as x, y axis respectively, a a constant and let $AN = x$ variable. $\Delta SNP \,|||\, \Delta SCT$ so that $\frac{SN}{NP} = \frac{SC}{CT}$ or $CT = \frac{SC \times NP}{SN}$.
Now, $SA = AN$ (by the property of the parabola since the tangent at $P(x, 2\sqrt{ax})$ has the equation $2y'\sqrt{ax} = 2a(x'+x)$ so that $x' = -x$ when $y' = 0$), $TB = BM$, $SC = a + x$.

Hence $$CT = \frac{(a+x) \times 2\sqrt{ax}}{2x} = \frac{\sqrt{a}(a+x)}{\sqrt{x}}$$

so that area of

$$\Delta STC = \frac{1}{2} SC \times CT = \frac{1}{2} \frac{(a+x)\sqrt{a}(a+x)}{\sqrt{x}} = \frac{\sqrt{a}}{2} \frac{(a+x)^2}{\sqrt{x}}.$$

Thus, we are to minimize $\frac{(a+x)^4}{x}$, or, equivalently, maximize $\frac{x}{(a+x)^4}$.

To this end, put $x = \frac{ab}{c}$. Then

$$\frac{x}{(a+x)^4} = \frac{\frac{ab}{c}}{(a+\frac{ab}{c})^4} = \frac{c^3}{a^3} \frac{b}{(b+c)^4}.$$

Thus we are to maximize

$$\frac{bc^3}{(b+c)^4} = \frac{b}{b+c} \times \frac{c^3}{(b+c)^3} = \left(1 - \frac{c}{b+c}\right)\left(\frac{c}{b+c}\right)^3 = (1-y)y^3$$

if $y = \frac{c}{b+c}$. i.e., to maximize $y^3 - y^4$. By solution to Problem 1, Chapter III, the maximum is reached when $y = \frac{c}{b+c} = \frac{3}{4}$ or $\frac{b+c}{c} = \frac{4}{3}$, or, $\frac{b}{c} = \frac{1}{3}$ or $x = \frac{ab}{c} = \frac{1}{3}a = \frac{a}{3}$.

Problem 4. AB is a diameter of a given circle. To find a point C on AB such that the rectangle formed by AC and the chord to the circle perpendicular to AB at C is a rectangle of maximum area.

Solution: Let the chord at C perpendicular to AB meet the circle at D, E. (Fig . 3.2). Then by secant theorem $CD^2 = AC.CB$. If $AC = x$, then $CD^2 = x(a-x)$, a

being the diameter of the circle. Thus, we are to maximize $AC \times 2CD$ or equivalently $AC.CD = x\sqrt{x(a-x)}$, or equivalently $x^3(a-x)$, i.e., $ax^3 - x^4 = a^4\left(\left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right)^4\right)$ or equivalently $y^3 - y^4$ where $y = \frac{x}{a}$.

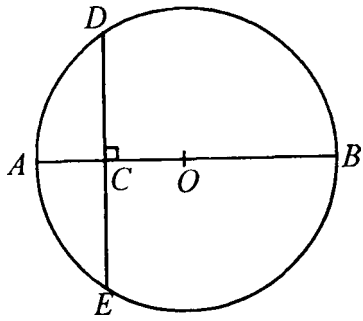


Fig. 3.2

By Problem 1, Chapter III, the solution is $y = \frac{x}{a} = \frac{3}{4}$ or $x = \frac{3a}{4}$.

Problem 5. To divide 12 into two summands so that the least multiplied by the cube of the greatest is a maximum.

Solution: If x is the greatest, then we are to maximize $x^3(a-x)$, when $a = 12$. i.e., $ax^3 - x^4$. As in Problem 4, the solution is $\frac{x}{a} = \frac{3}{4}$, i.e., $x = \frac{3 \times 12}{4} = 9$.

Problem 6. To inscribe an equilateral triangle of maximum area in a given circle.

Solution: Let ABC be an equilateral triangle inscribed in a given circle. (Fig. 3.3.). Let D be the midpoint of AC. Then $BD \perp AC$ by property of the circle and its diameter. If BD meets the circle again at E, then $AD \cdot DC = AD^2 = BD \cdot DE = x(2a-x)$ if a is the radius of the circle and $BD = x$, a variable. Area of $\Delta ABC = AD \cdot BD = x\sqrt{x(2a-x)}$.

Thus we are to maximize $x^3(2a-x) = 2ax^3 - x^4$ which is maximum by solution to Problem 5, Chapter III, when $\frac{x}{2a} = \frac{3}{4}$ or $x = \frac{3a}{2}$. Then

$AB^2 = BD^2 + AD^2 = x^2 + 2ax - x^2$

$= 2ax = 2a\frac{3a}{2} = 3a^2;$

so $AB = \sqrt{3}a$.

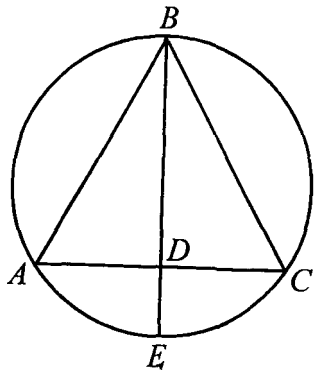


Fig. 3.3

Problem 7. To inscribe a parabolic sector symmetric about its axis and of maximum area in a given isosceles triangle.

Solution: Let GAF be the given isosceles triangle.(Fig. 3.4). Let the parabolic sector touch the equal sides GA, GF at H and M respectively. Let GD be the altitude from G of ΔGAF and let it meet the parabola at P and AF at D. Let K be the foot of the perpendicular from H on AD.

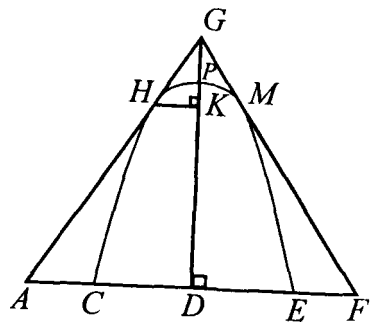


Fig. 3.4

Let the parabola meet AF at C, E. Then P is the vertex of the parabola. Let $AD = b$, $GD = a$, constants. Let $GP = x$. We know (See solution to Problem 3) that $GP = PK$ so that $GK = 2x$. By the property of the parabola $\frac{PK}{PD} = \frac{HK^2}{CD^2}$ (by the equation of the parabola). Also, by similar triangles $(\Delta sGHK,\ GAD)$, $\frac{GD}{AD} = \frac{GK}{HK}$, i.e.,

$\frac{a}{b} = \frac{2x}{HK}$, i.e., $HK = \frac{2bx}{a}$. Hence

$$\frac{x}{a-x} = \frac{4b^2\frac{x^2}{a^2}}{CD^2}$$

or

$$CD^2 = \frac{4b^2x^2(a-x)}{a^2x} = \frac{4b^2}{a^2}x(a-x).$$

or $CD = \frac{2b}{a}\sqrt{x(a-x)}$. Area of the parabolic sector

$$= \frac{2}{3}PD \times CD = \frac{2}{3}(a-x) \times \frac{2b}{a}\sqrt{x(a-x)} = \frac{4b}{3a}\sqrt{x}\sqrt{(a-x)^3}.$$

We are to maximize $x(a-x)^3$. Putting $a - x = y$, we are to maximize $y^3(a-y) = ay^3 - y^4$. The solution is $\frac{y}{a} = \frac{3}{4}$ or $y = \frac{3a}{4}$. So, $x = a - y = a - \frac{3a}{4} = \frac{a}{4}$.

Note: Use of calculus leads to a solution of a cubic equation which is tedious.

Problem 8. To find the parabolic sector of greatest area which can be cut from a given right circular cone.

Solution: Let ACD be a vertical cross section of the cone. (Fig. 3.5). Let the parabola be obtained by cutting by a plane parallel to AC. Let N be the point at which RM meets CD. Let the sector meet the circumference of the base in R, M. Let V be the vertex of the parabolic sector lying on the generator AD. VN is then the axis of the parabola. Let $AC = AD = b$, $CD = a$, constants. Let $DN = x$, variable. $\Delta VND \;|||\; \Delta ACD$ so that $\frac{AC}{VN} = \frac{DC}{DN}$, i.e., $\frac{b}{VN} = \frac{a}{x}$ or $VN = \frac{bx}{a}$. By secant theorem $RN^2 = CN \times ND = (a-x)x$, i.e., $RN = \sqrt{x(a-x)}$.

Area of the parabolic sector $= \frac{2}{3}\frac{bx}{a}\sqrt{ax - x^2}$ ($\frac{2}{3}$ rd the area of the parallelogram with the axis length and the ordinate as base and the distance between parallel sides, respectively). We are to maximize $x\sqrt{ax - x^2}$, or equivalently, $ax^3 - x^4$ which occurs when $\frac{x}{a} = \frac{3}{4}$ or $x = \frac{3a}{4}$. (See Solution of Problem 4, Chapter III).

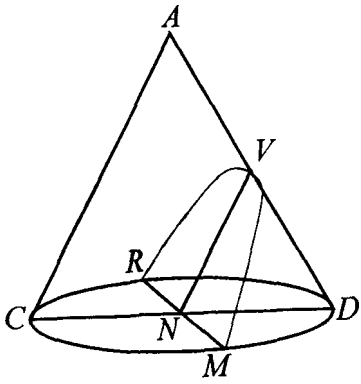


Fig. 3.5

Problem 9. The corner of a page of a book is turned back, so as just to reach the other edge of the page. Find when the part turned down has a minimum area.

Solution: (See Chapter II, Problem 20 and the figure therein). To recall $ABMN$ is the page which is folded so that the corner A falls at A' on MB. The crease formed is QP (see Fig. 3.6). Note that $Q\hat{A}P = Q\hat{A}'P = 90°$. Let $AB = a$, $AP = x$.

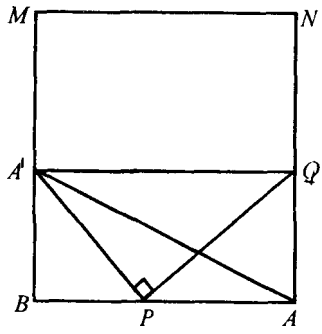


Fig. 3.6

By cosine formula

$$AA'^2 = AP^2 + PA'^2 - 2AP.PA' \cos A\hat{P}A'$$

$$= AP^2 + PA'^2 + 2AP.PA' \cos A'\hat{P}B$$

$$= AP^2 + PA'^2 + 2AP.BP = 2x^2 + 2x(a-x)$$

(since $AP = A'P$)

$$= 2ax.$$

$AA' = \sqrt{2ax}$. $AQ^2 = QP^2 - AP^2$. By the property of cyclic quadrilateral $APA'Q$; $A'A \times PQ = 2AQ \times AP$ noting that $AQ = A'Q$ too. Thus, $A'A^2 \times PQ^2 = 4AQ^2 \times AP^2 = 4(PQ^2 - AP^2) \times AP^2 = 4PQ^2 \times AP^2 - 4AP^4$. So, $4AP^4 = (4AP^2 - A'A^2)PQ^2$ i.e., $4x^4 = (4x^2 - 2ax) \times PQ$ or $PQ = \sqrt{\dfrac{2x^3}{2x-a}}$. Area of folded part $= \dfrac{1}{2}AP \times AQ$. Also $A'A \times PQ = 2AQ \times AP$. Hence the area $\dfrac{1}{4}A'A \times PQ = \dfrac{1}{4}\sqrt{2ax}\sqrt{\dfrac{2x^3}{2x-a}}$. We are to minimize therefore $\sqrt{\dfrac{x^4}{2x-a}}$. Equivalently, we are to maximize

$$\frac{2x-a}{x^4} = \frac{2}{x^3} - \frac{a}{x^4} = \frac{2^4}{a^3}\left[\frac{a^3}{(2x)^3} - \frac{a^4}{(2x)^4}\right].$$

Thus, it is enough to maximize $y^3 - y^4$ where $y = \dfrac{a}{2x}$. The solution is $\dfrac{a}{2x} = \dfrac{3}{4}$ or $x = \dfrac{2a}{3}$.

Problem 10. To find x such that $mx^4 - x^5$ is maximum for fixed $m > 0$.

Solution: Write $mx^4 - x^5 = r$ or $x^5 - mx^4 + r = 0$. This fifth degree equation has one real root, 3 real roots or all 5 real roots. Since complex roots α occur in conjugate pairs with $(x-\alpha)(x-\bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha.\bar{\alpha} = x^2 - (2\mathrm{Re}\bar{\alpha})x + |\alpha|^2$ there is a quadratic polynomial which is a factor of the quintic

(5th degree) polynomial $x^5 - mx^4 + n$. Let this quadratic polynomial multiplied by the monomial corresponding to the real root be $x^3 + ax^2 + bx + c$. Dividing $x^5 - mx^4 + r$ by this third degree polynomial, we have

$$\begin{aligned} & x^5 - mx^4 + r \\ = \; & (x^3 + ax^2 + bx + c)(x^2 - (a+m)x + a^2 + am - b) \\ & + \{(ab + bm - c) - (a^3 + a^2m - ab)\}\, x^2 \\ & + \{(ac + cm) - (a^2b + abm - b^2)\}\, x + r - c(a^2 + am - b). \end{aligned}$$

x corresponding to maximum r thus satisfies

$$x^2 - (a+m)x + a^2 + am - b = 0 \tag{1}$$

$$ab + bm - c = a^3 + a^2m - ab \tag{2}$$

$$(ac + cm) = a^2b + abm - b^2 \tag{3}$$

$$r = c(a^2 + am - b). \tag{4}$$

From (2) and (3)

$$\begin{aligned} a^3 + a^2m - ab &= ab + bm - c = ab + bm - \frac{a^2b + abm - b^2}{a+m} \\ &= \frac{a^2b + abm + abm + bm^2 - a^2b - abm + b^2}{a+m} \\ &= \frac{abm + bm^2 + b^2}{a+m} \end{aligned}$$

so that

$$abm + bm^2 + b^2 = (a+m)(a^3 + a^2m - ab) = a^4 + 2ma^3 + m^2a^2 - a^2b - mab$$

or $\quad b^2 + (2ma + m^2 + a^2)b = a^4 + 2ma^3 + m^2a^2 = a^2(a+m)^2$

i.e., $b^2 + (a+m)^2 b + \dfrac{(a+m)^4}{4} = a^2(a+m)^2 + \dfrac{(a+m)^4}{4}$

i.e., $\left\{b + \dfrac{(a+m)^2}{2}\right\}^2 = (a+m)^2 \left\{\dfrac{4a^2 + (a+m)^2}{4}\right\}$

i.e., $b = \dfrac{-(a+m)^2}{2} \pm \dfrac{(a+m)}{2}\sqrt{4a^2 + (a+m)}$

$= \dfrac{-(a+m)^2 \pm (a+m)\sqrt{4a^2 + (a+m)}}{2}$.

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the roots of $x^5 - mx^4 + r = 0$, the sum of all the products of two of the roots taken at a time is 0. Also, at least one of the roots is real and the product of the roots is negative.

In fact, there is one real root since $x^5 - mx^4 < 0 (> 0)$ according as $x < m (x > m)$ for $x > 0$. $x^5 - mx^4$ vanishes for $x = 0$ and $x = m$. Thus, if we know that $x^5 - mx^4$ decreases from $x = 0$ to some stage of x (reaching a value $-r$) and then increases to 0 between this stage and $x = m$, the graph of $y = x^5 - mx^4 + r$ is obtained by lifting the graph of $y = x^5 - mx^4$ bodily vertically by a distance r so that the graph of $y = x^5 - mx^4 + r$ touches the x-axis after this lifting. (See Fig. 3.7).

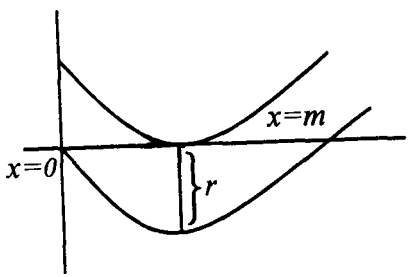


Fig. 3.7

Thus $x^5 - mx^4 + r = 0$ has precisely one real root which is negative. Let the other roots be $\beta \pm i\gamma, \delta \pm i\varepsilon$. Then, if $-\alpha$ is the negative root, since the coefficient of x^3 in $x^5 - mx^4 + r = 0$ is zero $-2\alpha(\beta + \delta) + |\beta + i\gamma|^2 + |\delta + i\varepsilon|^2 = 0$. It follows

that one of $-2\alpha\beta + |\beta + i\gamma|^2$ or $-2\alpha\delta + |\delta + i\varepsilon|^2$ should be positive and the other negative. We take as the roots of $x^3+ax^2+bx+c=0$ either $-\alpha, \beta \pm i\gamma$ or $-\alpha, \delta \pm i\varepsilon$ depending on whether $-2\alpha\beta + |\beta + i\gamma|^2$ or $-2\alpha\delta + |\delta + i\varepsilon|^2$ is negative. Thus b is positive so that we ignore the negative alternative for the value of b. Note that $c = \alpha|\beta + i\gamma|^2$ or $-\alpha|\delta + i\varepsilon|^2$ is positive. From (1),(4), we then have $x^2 - (a+m)x = -\dfrac{r}{c}$

i.e., $$x^2 - (a+m)x + \frac{(a+m)^2}{4} = \frac{(a+m)^2}{4} - \frac{r}{c}$$

i.e., $$\frac{r}{c} = \frac{(a+m)^2}{4} - \left(x - \frac{a+m}{2}\right)^2.$$

Thus r is maximum when $x = \dfrac{a+m}{2}$.

Further, the maximum value of $\dfrac{r}{c}$ is $\dfrac{(a+m)^2}{4} = a^2 + am - b$.

So, $$\frac{(a+m)^2}{4} = a(a+m) + \frac{(a+m)^2 - (a+m)\sqrt{4a^2 + (a+m)^2}}{2}$$

$$= \frac{2a(a+m) + (a+m)^2 - (a+m)\sqrt{4a^2 + (a+m)^2}}{2}.$$

or

$$\frac{a+m}{2} = 2a + (a+m) - \sqrt{4a^2 + (a+m)^2}$$

i.e., $$a + m = 6a + 2m - 2\sqrt{4a^2 + (a+m)^2}$$

i.e., $$5a + m = 2\sqrt{4a^2 + (a+m)^2}$$

i.e., $$25a^2 + 10am + m^2 = 16a^2 + 4a^2 + 8am + 4m^2$$

i.e., $$5a^2 + 2am - 3m^2 = 0, \quad \text{i.e., } (5a - 3m)(a+m) = 0.$$

Therefore $a = \dfrac{3m}{5}$ since $a \neq -m$. Note that the sum of all the five roots of the quintic is m and if $a = -m$, it will follow that the sum of the three roots of $x^3 + ax^2 + bx + c = 0$

is itself m. Thus, either $\beta = 0$ or $\delta = 0$, in the earlier notation, and $x^5 - mx^4 + r \equiv (x^3 + ax^2 + bx + c)(x^2 + \gamma^2)$ (or $(x^3 + ax^2 + bx + c)(x^2 + \delta^2)$). On comparison of coefficients it follows then that $b = \gamma = 0 = r$ which is absurd. So, $x = \dfrac{a+m}{2} = \dfrac{\frac{3m}{5} + m}{2} = \dfrac{4m}{5}$. For $m = 1$, $x = \dfrac{4}{5}$.

Note: It suffices to consider $x^5 - x^4$ in the place of $x^5 - mx^4$ since $x^5 - mx^4 = m^5 \left(\left(\frac{x}{m}\right)^5 - \left(\frac{x}{m}\right)^4 \right)$.

Problem 11. To find $x > 0$ such that $mx^3 - x^5$ is maximum for fixed $m > 0$.

Solution: Similar to the solution of Problem 10 we have here the equation $x^5 - mx^3 + r = 0$. Here too, by the nature of the function $x^5 - mx^3$, there is only one real root for the above quintic equation and this root is negative, say $-\alpha, \alpha > 0$. Let the cubic polynomial whose roots are $-\alpha$ and two roots of the quintic which are complex conjugates of each other (to be properly chosen) be $x^3 + ax^2 + bx + c$. Then c is positive.

Dividing $x^5 - mx^3 + r$ by this polynomial, we have

$$\begin{aligned} x^5 - mx^3 + r &= (x^3 + ax^2 + bx + c)(x^2 - ax + a^2 - b - m) \\ &\quad + \{(ab - c) - (a^3 - ab - am)\} x^2 \\ &\quad + \{ca - (a^2 b - b^2 - bm)\} x \\ &\quad + r - (ca^2 - bc - cm). \end{aligned}$$

so that we have

$$x^2 - ax + a^2 - b - m = 0 \tag{1}$$

$$ab - c = a^3 - ab - am \tag{2}$$

$$ca = a^2b - b^2 - bm \qquad \text{or} \qquad c = \frac{a^2b - b^2 - bm}{a} \tag{3}$$

$$\frac{r}{c} = a^2 - b - m. \tag{4}$$

From (2) and (3)

$$a(a^2 - b - m) = ab - c = ab - \frac{a^2b - b^2 - bm}{a}$$

$$= \frac{b^2 + bm}{a}.$$

So, $b^2 + bm = a^2(a^2 - b - m)$, i.e., $b^2 + b(m + a^2) = a^2(a^2 - m)$

i.e., $b^2 + b(m + a^2) + \dfrac{(m + a^2)^2}{4} = a^2(a^2 - m) + \dfrac{(m + a^2)^2}{4}$

i.e., $$\left(b + \frac{m + a^2}{2}\right)^2 = \frac{4a^4 - 4a^2m + m^2 + 2ma^2 + a^4}{4}$$

$$= \frac{5a^4 - 2ma^2 + m^2}{4}.$$

Thus

$$b = -\frac{m + a^2}{2} \pm \frac{1}{2}\sqrt{5a^4 - 2ma^2 + m^2}. \tag{5}$$

We choose the roots of $x^3 + ax^2 + bx + c = 0$ in such a way that b is negative. i.e., if $-\alpha, \beta \pm i\gamma, \delta \pm i\varepsilon$ are the roots of the quintic we can take for the roots of $x^3 + ax^2 + bx + c = 0$ either $-\alpha, \beta \pm i\gamma$ or $-\alpha, \delta \pm i\varepsilon$, such that

$$-\alpha(\beta + i\gamma) - \alpha(\beta - i\gamma) + (\beta + i\gamma)(\beta - i\gamma) = b$$

or $$-\alpha(\delta + i\varepsilon) - \alpha(\delta - i\varepsilon) + (\delta + i\varepsilon)(\delta - i\varepsilon) = b$$

is negative.

i.e., $$-2\alpha\beta + \beta^2 + \gamma^2 < 0 \quad \text{or} \quad -2\alpha\delta + \delta^2 + \varepsilon^2 < 0.$$

If both of them are positive, by adding both

$$-2\alpha(\beta+\delta)+\beta^2+\gamma^2+\delta^2+\varepsilon^2>0 \qquad (6)$$

But, $-\alpha, \beta\pm i\gamma, \delta\pm i\varepsilon$ being roots of $x^5-mx^3+r=0$

$$-\alpha(\beta+i\gamma)-\alpha(\beta-i\gamma)-\alpha(\delta+i\varepsilon)-\alpha(\delta-i\varepsilon)$$

$$+(\beta+i\gamma)(\beta-i\gamma)+(\beta+i\gamma)(\delta+i\varepsilon)+(\beta+i\gamma)(\delta-i\varepsilon)$$

$$+(\beta-i\gamma)(\delta+i\varepsilon)+(\beta-i\gamma)(\delta-i\varepsilon)+(\delta+i\varepsilon)(\delta-i\varepsilon)$$

$$=-m$$

and

$-\alpha+\beta+i\gamma+\beta-i\gamma+\delta+i\varepsilon+\delta-i\varepsilon=-\alpha+2\beta+2\delta=0$

i.e., $\alpha=2(\beta+\delta)$.

The former equation gives

$$-\alpha(2\beta+2\delta)+\beta^2+\gamma^2+(\beta+i\gamma)2\delta+(\beta-i\gamma)2\delta+\delta^2+\varepsilon^2=-m$$

i.e., $\quad -\alpha(2\beta+2\delta)+4\beta\delta+\beta^2+\gamma^2+\delta^2+\varepsilon^2=-m<0. \quad (7)$

From (6)and (7) it follows that $\beta\delta<0$ while $\alpha=2(\beta+\delta)>0$.

Also, the sum of the products of four roots taken at a time for the quintic $x^5-mx^3+r=0$ is 0.

i.e., $\qquad (\beta^2+\gamma^2)(\delta^2+\varepsilon^2)-2\alpha\delta(\beta^2+\gamma^2)-2\alpha\beta(\delta^2+\varepsilon^2)=0$

i.e., $\qquad (\beta^2+\gamma^2-2\alpha\beta)(\delta^2+\varepsilon^2-2\alpha\delta)-4\alpha^2\beta\delta=0$

i.e., $\qquad (\beta^2+\gamma^2-2\alpha\beta)(\delta^2+\varepsilon^2-2\alpha\delta)=4\alpha^2\beta\delta$.

If both of the two factors on the $L.H.S$ are positive then $\beta\delta>0$. There is a contradiction. It therefore follows that one of $-2\alpha\beta+\beta^2+r^2$, $-2\alpha\delta+\delta^2+\varepsilon^2$ is negative. Thus b is

negative by choice and we ignore presently the $+$ alternative in the value for b in (5). From (1), $x^2 - ax = -\dfrac{r}{c}$

i.e., $\qquad \dfrac{r}{c} + \left(x - \dfrac{a}{2}\right)^2 = \dfrac{a^2}{4}$,

i.e., $\qquad \dfrac{r}{c} = \dfrac{a^2}{4} - (x - \dfrac{a}{2})^2$, c is positive since

$c = -\alpha(\beta + i\gamma)(\beta - i\gamma)(\delta + i\varepsilon)(\delta - i\varepsilon) = -\alpha(\beta^2 + \gamma^2)(\delta^2 + \varepsilon^2)$..

So $\dfrac{r}{c}$ and so r is maximum when $x = \dfrac{a}{2}$. Then $\dfrac{r}{c} = \dfrac{a^2}{4}$.

i.e., $\qquad a^2 - b - m = \dfrac{a^2}{4}$ from (4)

i.e., $\qquad \dfrac{a^2}{4} = \dfrac{2a^2 - 2m + m + a^2 + \sqrt{5a^4 - 2ma^2 + m^2}}{2}$

i.e., $\qquad a^2 = 6a^2 - 2m + 2\sqrt{5a^4 - 2ma^2 + m^2}$

i.e., $\qquad 2m - 5a^2 = 2\sqrt{5a^4 - 2ma^2 + m^2}$

i.e., $\qquad 4m^2 - 20mr^2 + 25a^4 = 20a^4 - 8ma^2 + 4m^2$

i.e., $\qquad 5a^4 = 12ma^2$,

i.e., $\qquad a^2 = \dfrac{12}{5}m$ So, $x = \dfrac{a}{2} = \sqrt{\dfrac{12}{5}m \times \dfrac{1}{2}} = \sqrt{\dfrac{3m}{5}}$.

Note: The alternative $+$ in the value of b in (5) also gives the same value for a.

Problem 12. To find $x > 0$ such that $mx^2 - x^5$ is maximum, given $m > 0$.

Solution We are to find the real root of $x^5 - mx^2 + r = 0$, where $r = \max_{x>0}\left(mx^2 - x^5\right)\cos$.

Here too the real root is negative, say $-\alpha, \alpha > 0$ say. We adhere to the notation in the solution to Problem 11.

Let $-\alpha$ and $\beta \pm i\gamma$ be the roots of $x^3 + ax^2 + bx + c = 0$. Then $x^5 - mx^2 + r = (x^3 + ax^2 + bx + c)(x^2 - ax + a^2 - b) + x^2\{(ab - c - m) - (a^3 - ab)\} + x\{ac - (a^2b - b^2)\} + r - c(a^2 - b)$.

Thus

$$x^2 - ax + a^2 - b = 0 \tag{1}$$

$$ab - c - m = a^3 - ab \tag{2}$$

$$ac = a^2b - b^2 \tag{3}$$

$$r = c(a^2 - b) \tag{4}$$

From (3) $c = \dfrac{a^2b - b^2}{a}$ and so

$$ab - m = a^3 - ab + \frac{a^2b - b^2}{a} = \frac{a^4 - a^2b + a^2b - b^2}{a}$$

i.e., $$a^2b - am = a^4 - b^2 \text{ or } b^2 + a^2b = a^4 + am$$

i.e., $$\left(b + \frac{a^2}{2}\right)^2 = \frac{a^4}{4} + a^4 + am = \frac{5}{4}a^4 + am$$

$$b = -\frac{a^2}{2} \pm \frac{1}{2}\sqrt{5a^4 + 4am}.$$

Taking the first alternative value for b, from (1) and (4) $x^2 - ax + a^2 - b = x^2 - ax + \dfrac{r}{c} = 0$ implies $\dfrac{r}{c} = \dfrac{a^2}{4} - (x - \dfrac{a}{2})^2$. c being positive (cf. solution of Problem 11), r is maximum when $x = \dfrac{a}{2}$ and the maximum value is given by $\dfrac{a^2}{4} = \dfrac{r}{c}$.

Thus, from (4)

$$\frac{a^2}{b} = \frac{r}{c} = a^2 - b = a^2 + \frac{a^2 - \sqrt{5a^4 + 4am}}{2}$$

i.e., $$\frac{3a^2 - \sqrt{5a^4 + 4am}}{2} = \frac{a^2}{4},$$

i.e., $5a^2 = 2\sqrt{5a^4 + am}$.

i.e., $25a^4 = 20a^4 + 16am$, i.e., $5a^4 = 16am$, i.e., $5a^3 = 16m$,

i.e., $a^3 = \frac{16m}{5}$, $x = \frac{a}{2}$, $x^3 = \frac{a^3}{8} = \frac{2m}{5}$, $x = \sqrt[3]{\frac{2m}{5}}$.

Note: The negative alternative for the value of b gives the same value for a.

Problem 13. Find $x > 0$ such that $mx - x^5$ is maximum, given $m > 0$.

Solution: If r is the maximum value sought, we are to consider in this case the equation $x^5 - mx + r = 0$. One of the roots is real and negative and if the cubic equation which has this root and two other roots of the quintic (complex non real roots) is $x^3 + ax^2 + bx + c = 0$, then

$$\begin{aligned} x^5 - mx + r \equiv\ & (x^3 + ax^2 + bx + c)(x^2 - ax + a^2 - b) \\ & + x^2\{(ab - c) - (a^3 - ab)\} \\ & + x\{(ac - m) - (a^2b - b^2)\} \\ & + r - c(a^2 - b), \end{aligned}$$

so that

$$c(a^2 - b) = r, \text{ i.e., } a^2 - b = \frac{r}{c} \tag{2}$$

$$ab - c = a^3 - ab \tag{3}$$

$$ca - m = a^2b - b^2. \tag{4}$$

Besides

$$x^2 - ax + a^2 - b = 0. \tag{1}$$

From (1)

$$x^2 - ax + \frac{r}{c} = 0$$

i.e., $$\frac{r}{c} = \frac{a^2}{4} - \left(x - \frac{a}{2}\right)^2 .$$

c being positive r is maximum when $x = \frac{a}{2}$ and the maximum value of r is given by $\frac{r}{c} = \frac{a^2}{4}$. Using (3) and (4)

$$a^3 - ab = ab - c = ab - \frac{a^2b - b^2 + m}{a} = \frac{b^2 - m}{a}$$

i.e., $b^2 + a^2b = a^4 + m$, i.e., $b^2 + a^2b + \frac{a^4}{4} = \frac{5a^4}{4} + m$

i.e., $(b + \frac{a^2}{2})^2 = \frac{5a^4 + 4m}{4}, b = -\frac{a^2}{2} \pm \frac{\sqrt{5a^4 + 4m}}{2}$.

Now for maximum value r

$$\frac{r}{c} = \frac{a^2}{4} = a^2 - b = a^2 + \frac{a^2 - \sqrt{5a^2 + 4m}}{2}$$

$$= \frac{3a^2 - \sqrt{5a^4 + 4m}}{2}$$

using the first alternative. Hence

$$a^2 = 6a^2 - 2\sqrt{5a^4 + 4m}$$

i.e., $$5a^2 = 2\sqrt{5a^4 + 4m}$$

i.e., $$25a^4 = 20a^4 + 16m$$

i.e., $$5a^4 = 16m$$

i.e., $$a^4 = \frac{16}{5}m,$$

i.e., $$a = \sqrt[4]{\frac{16m}{5}}$$

So, the corresponding value of

$$x = \frac{a}{2} = \frac{1}{2}\sqrt[4]{\frac{16m}{5}} = \sqrt[4]{\frac{16m}{16 \times 5}} = \sqrt[4]{\frac{m}{5}}.$$

Note: The negative alternative for the value of b also gives the same value for a provided a is assumed positive. This is positive since (the quintic has two real roots, one positive and one negative) If $-\alpha$ (twice repeated $\alpha > 0)\beta, \gamma \pm i\delta$ are the roots and if $-\alpha + 2\gamma, \beta + 2\gamma, -2\alpha + \beta$ are all positive, then $-3\alpha + 2\beta + 4\gamma$ is positive while the sum of the roots of the quintic $-2\alpha + \beta + 2\gamma = 0$, which implies $-\alpha$ is positive, a contradiction.

Problem 14. To find $x > 0$ such that $mx^5 - x^6$ is maximum for a given $m > 0$.

Solution: The equation involved here is $x^6 - mx^5 + r = 0$ which has only one negative real root. If $x^4 + ax^3 + bx^2 + cx + d$ is the polynomial having four of the roots of the quintic as roots (complex conjugate roots having been taken together).

$$
\begin{aligned}
& x^6 - mx^5 + r \\
= \; & (x^4 + ax^3 + bx^2 + cx + d)(x^2 - (a+m)x + a^2 + am - b) \\
& + x^3\left\{(ab + bm - c) - (a^3 + a^2m - ab)\right\} \\
& + x^2\left\{(ca + cm - d) - (a^2b + abm - b^2)\right\} \\
& + x\left\{x(ad + dm) - (ca^2 + cam - bc)\right\} \\
& + r - d(a^2 + am - b).
\end{aligned}
$$

Thus
$$x^2 - (a+m)x + a^2 + am - b = 0 \qquad (1)$$

and
$$r = d(a^2 + am - b); \qquad \text{i.e.,} \quad \frac{r}{d} = a^2 + am - b \qquad (2)$$

Also, $$ab + bm - c = a^3 + a^2m - ab \tag{3}$$

$$ca + m - d = a^2b + abm - b^2. \tag{4}$$

$$ad + dm = ca^2 + acm - bc, \quad \text{i.e.,} \quad d = \frac{ca^2 + acm - bc}{a + m} \tag{5}$$

So, from (5)

$$\begin{aligned} ca + cm - d &= c(a + m) - d = c(a + m) - \frac{c(a^2 + am - b)}{a + m} \\ &= \frac{c(a + m)^2 - c(a^2 + am - b)}{a + m} \\ &= \frac{cm(a + m) + bc}{a + m} \end{aligned}$$

From (4) $ab(a + m) - b^2 = \dfrac{cm(a + m) + bc}{a + m}$

Hence $c = \dfrac{b(a + m)\{a(a + m) - b\}}{m(a + m) + b}$.

From (3) $$\begin{aligned} ab + bm - c &= b(a + m) - \frac{b(a + m)\{a(a + m) - b\}}{m(a + m) + b} \\ &= \frac{mb(a + m)^2 + b^2(a + m) - ab(a + m)^2 + b^2(a + m)}{m(a + m) + b} \\ &= \frac{2b^2(a + m) + b(a + m)^2(m - a)}{m(a + m) + b} \\ &= a^2(a + m) - ab \end{aligned}$$

i.e., $$\begin{aligned} &2b^2(a + m) + b(a + m)^2(m - a) \\ &= ma^2(a + m)^2 + ba^2(a + m) - abm(a + m) - ab^2 \\ &= ma^2(a + m)^2 + ab(a + m)(a - m) - ab^2 \end{aligned}$$

i.e.,

$$b^2\{2(a+m)+a\}+b\{(a+m)^2(m-a)-a(a+m)(a-m)\}$$
$$= ma^2(a+m)^2$$

i.e., $b^2+b\dfrac{(2a+m)(a+m)(m-a)}{3a+2m}=\dfrac{ma^2(a+m)^2}{3a+2m}$

i.e.,

$$b = -\frac{(2a+m)(a+m)(m-a)}{2(3a+2m)}$$
$$\pm\frac{1}{2(3a+2m)}\{4ma^2(a+m)^2(3a+2m)+(2a+m)^2$$
$$\times(a+m)^2(m-a)^2\}^{1/2}.$$

i.e., $b = \Big\{-(a+m)(2a+m)(m-a)$
$$\pm\sqrt{(4a^4+8a^3m+5a^2m^2+2am^3+m^4)\times(a+m)^2}\Big\}$$
$$\Big/\Big\{2(3a+2m)\Big\}$$
$$= \frac{-(a+m)(2a+m)(m-a)+(a+m)^2\sqrt{4a^2+m^2}}{2(3a+2m)}$$

(Taking the positive alternative)

From (2), taking the above value of b

$$a(a+m)-b$$
$$= \Big\{2a(3a+2m)(a+m)-(a+m)(2a+m)(a-m)$$
$$-(a+m)^2\sqrt{4a^2+m^2}\Big\}\Big/\Big\{2(3a+2m)\Big\}$$

Now,

$$(a+m)(2a+m)(a-m) = 2a^3 + a^2m - 2am^2 - m^3.$$

$$\begin{aligned} 2a(3a+2m)(a+m) - (a+m)(2a+m)(a-m) &= 4a^3 + 9a^2m + 6am^2 + m^3 \\ &= (a^2 + 2ma + m^2)(4a+m) \\ &= (a+m)^2(4a+m). \end{aligned}$$

Thus

$$a(a+m) - b = \frac{(a+m)^2(4a+m) - (a+m)^2\sqrt{4a^2+m^2}}{2(3a+2m)}$$

Now

$$x^2 - (a+m)x + \left(\frac{a+m}{2}\right)^2 + a^2 + am - b = \left(\frac{a+m}{2}\right)^2$$

i.e.,
$$\frac{r}{d} = \left(\frac{a+m}{2}\right)^2 - \left(x - \frac{a+m}{2}\right)^2$$

Note that $mx^5 - x^6$ vanishes at $x = 0$, $x = m$ and increases starting from 0 upto a stage and thereafter decreases to 0.

Thus $x^6 - mx^5 + r = 0$ has only one real negative root. Thus the biquadratic $x^4 + ax^3 + bx^2 + cx + d$ has all its roots nonreal. Thus d is positive.

Hence r is maximum when $x = \dfrac{a+m}{2}$ and the maximum value r is given by $\dfrac{r}{d} = \left(\dfrac{a+m}{2}\right)^2$.

Now,

$$\frac{r}{d} = a^2 + am - b = \frac{(a+m)^2(4a+m) - (a+m)^2\sqrt{4a^2+m^2}}{2(3a+2m)}$$

So, $\left(\frac{a+m}{2}\right)^2 = \frac{(a+m)^2\left((4a+m)-\sqrt{4a^2+m^2}\right)}{2(3a+2m)}$

i.e., $3a+2m = 2(4a+m) - 2\sqrt{4a^2+m^2}$

i.e., $5a = 2\sqrt{4a^2+m^2}$

i.e., $25a^2 = 16a^2 + 4m^2$

i.e., $9a^2 = 4m^2,$

i.e., $a = \frac{2}{3}m.$

So, $x = \frac{\frac{2}{3}m+m}{2} = \frac{5}{6}m.$

Note: Here too the positive alternative gives the same value for a assumed to be positive. We can check that the four complex roots are such that sum of the roots of the biquadratic is negative. For, if $\beta \pm i\gamma, \delta \pm i\varepsilon$ are the complex roots and the repeated real root is $-\alpha(\alpha > 0)$, let $-2\alpha+2\beta, -2\alpha+2\gamma, 2\beta+2\gamma$ be all positive. Then

$$-m = -2\alpha + 2\beta + 2\gamma = \frac{(-2\alpha+2\beta-2\alpha+2\gamma+2\beta+2\gamma)}{2}$$

is also positive which is a contradiction.

Problem 15. To find $x > 0$ such that $mx^4 - x^6$ is maximum for given $m > 0$.

Solution: Putting $x^2 = y$, the problem reduces to finding the value of $y > 0$ for which $my^2 - y^3$ is maximum. This is solved as a special case of Problem 2 of Chapter II. The solution is $y = \frac{2m}{3} = x^2$ and so $x = \sqrt{\frac{2m}{3}}$.

Problem 16. To find $x > 0$ such that $mx^3 - x^6$ is maximum for a given $m > 0$.

Solution: Putting $x^3 = y$ the problem reduces to that of finding $y > 0$ such that $my - y^2$ is maximum. By Problem 1 of Chapter II, the solution is $y = \frac{m}{2} = x^3$, i.e., $x = \sqrt[3]{\frac{m}{2}}$.

Problem 17. To find $x > 0$ such that $mx^2 - x^6$ is maximum for a given $m > 0$.

Solution: Putting $x^2 = y$, the problem reduces to finding $y > 0$ such that $my - y^3$ is maximum. By Problem 1 of Chapter II the solution is $y = \sqrt{\frac{m}{3}} = x^2$ so that $x = \sqrt[4]{\frac{m}{3}}$.

Problem 18. To find $x > 0$ such that $mx - x^6$ is maximum for a given $m > 0$.

Solution: Here the equation to be considered is $x^6 - mx + r = 0$. $x^6 - mx = 0$ has two real roots $x = 0$, $x = \sqrt[5]{m}$. $x^6 - mx + r = 0$ also has a repeated real root which is positive since $r > 0$. Let the biquadratic having four roots of the equation $x^6 - mx + r = 0$ be $x^4 + ax^3 + bx^2 + cx + d = 0$. Then

$$\begin{aligned} x^6 - mx + r \\ = \; & (x^4 + ax^3 + bx^2 + cx + d)(x^2 - ax + a^2 - b) \\ & + x^3\{(ab - c) - (a^3 - ab)\} + x^2\{(ac - d) - (a^2b - b^2)\} \\ & + x\{(ad - m) - (a^2c - bc)\} + r - d(a^2 - b). \end{aligned}$$

Then

$$x^2 - ax + a^2 - b = 0 \qquad (1)$$

$$d(a^2 - b) = r \text{ or } \frac{r}{d} = a^2 - b \tag{2}$$

Also

$$ab - c = a^3 - ab \tag{3}$$

$$ac - d = a^2b - b^2 \tag{4}$$

$$ad - m = ca^2 - bc. \tag{5}$$

From (5), $d = \dfrac{ca^2 - bc + m}{a}$. So

$$ac - d = ac - \frac{ca^2 - bc + m}{a} = \frac{bc - m}{a} = a^2b - b^2.$$

Hence

$$c = \frac{a^3b - ab^2 + m}{b}.$$

Therefore

$$ab - c = ab - \frac{a^3b - ab^2 + m}{b} = \frac{2ab^2 - a^3b - m}{b} = a^3 - ab \quad \text{by (3)}$$

So, $2ab^2 - a^3b - m = a^3b - ab^2$, so that $3ab^2 - 2a^3b = m$;
$3a\left(b - \dfrac{a^2}{3}\right)^2 = m + 3a \cdot \dfrac{a^4}{9} = m + \dfrac{a^5}{3}$.

$$\begin{aligned}
\text{Thus} \quad b &= \frac{a^2}{3} \pm \sqrt{\frac{1}{3a}(m + \frac{a^5}{3})} \\
&= \frac{a^2}{3} \pm \frac{1}{3}\sqrt{\frac{3m + a^5}{a}} \\
&= \frac{a^2}{3} \pm \frac{1}{3a}\sqrt{a^6 + 3am} \\
&= \frac{a^3 \pm \sqrt{a^6 + 3am}}{3a}.
\end{aligned}$$

Hence $a^2 - b = a^2 - \dfrac{a^3 \pm \sqrt{a^6 + 3am}}{3a} = \dfrac{2a^3 - \sqrt{a^6 + 3am}}{3a}$

taking at present the positive alternative in the value of b.

Hence, by (2), $\dfrac{r}{d} = \dfrac{2a^3 - \sqrt{a^6 + 3am}}{3a} = a^2 - b$, this being the maximum value of $\dfrac{r}{d}$ as derived from (1), noting that d can be chosen to be positive. For, if $\alpha(> 0), \beta \pm i\gamma, \delta \pm i\varepsilon$ with α repeated twice are the roots of $x^6 - mx + r = 0$, then the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ are $\alpha, \alpha, \beta \pm i\gamma$; or $\alpha, \alpha, \delta \pm i\varepsilon$; or $\beta \pm i\gamma, \delta \pm i\varepsilon$ so that the product of the four roots of the biquadratic is positive.

From (1) $x^2 - ax + \dfrac{r}{d} = 0$. So $x = \dfrac{a}{2} + \sqrt{\dfrac{a^2}{4} - \dfrac{r}{d}}$.

Hence, when r is maximum

$$\frac{a^2}{4} = \frac{r}{d} = a^2 - b = \frac{2a^3 - \sqrt{a^6 + 3am}}{3a}.$$

Therefore, $3a^3 = 8a^3 - 4\sqrt{a^6 + 3am}$;

i.e., $5a^3 = 4\sqrt{a^6 + 3am}$;

i.e., $25a^6 = 16a^6 + 48am$; i.e., $9a^6 = 48am$; i.e., $a^5 = \frac{16}{3}m$.

So, $x^5 = \left(\dfrac{a}{2}\right)^2 = \dfrac{a^5}{32} = \dfrac{m}{6}$ or $x = \sqrt[5]{\frac{m}{6}}$.

CHAPTER IV

Problems of maxima and minima in which two or more variable quantities are used

Procedure: If there are two variables, under the conditions given (maxima or minima), find the value of each in terms of the other when the condition is satisfied. Thereby we get two equations comparing which we get the values of the two variables in terms of constants involved which satisfy the given condition. If there are three variables, equations satisfying the condition expressing each variable in terms of the other two (three in number) are obtained. Solving these equations we obtain the values of the variables satisfying the given condition. This procedure applies to more number of variables too.

Note that each equation will give infinite number of values but if all the equations were to hold, all the equations need to be satisfied simultaneously. In each equation we have to consider both the maximum and minimum.

Problem 1. To inscribe the greatest rectangular parallelopiped in a given ellipsoid.

Solution: Let the equation of the ellipsoid referred to its axis be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The rectangular parallelopiped has its faces parallel to the coordinate planes. Let the corner of the parallelopiped lying in the positive octant have coordinates (x, y, z). Then its sides are $2x$, $2y$, $2z$ and

hence its volume is $8xyz$. Now $z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$ so that $64x^2y^2z^2 = 64c^2x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$ or $64x^2y^2z^2 = \frac{64c^2}{a^2b^2}\,x^2y^2(a^2b^2 - b^2x^2 - a^2y^2)$. i.e., we are to maximize $a^2b^2x^2y^2 - b^2x^4y^2 - a^2x^2y^4$. Treating x as constant and y as variable,

$$a^2b^2x^2y^2 - b^2x^4y^2 - a^2x^2y^4 = a^2x^2\left[\frac{a^2b^2y^2 - b^2x^2y^2}{a^2} - y^4\right].$$

i.e., we are to maximize $\frac{a^2b^2y^2 - b^2x^2y^2}{a^2} - y^4$. This gives rise to $y^4 - \frac{a^2b^2 - b^2x^2}{a^2}y^2 + r = 0$. Completing the square, we have for the value of y^2 which makes r maximum $y^2 = \frac{a^2b^2 - b^2x^2}{2a^2}$.

Considering now y as constant we have similarly the value of x^2 which makes $a^2b^2x^2y^2 - b^2x^4y^2 - a^2x^2y^4$ maximum given by $x^2 = \frac{a^2b^2 - a^2y^2}{2b^2}$. Solving for y^2, we have $y^2 = \frac{a^2b^2 - 2b^2x^2}{a^2} = \frac{a^2b^2 - b^2x^2}{2a^2}$ (from the value of y^2 obtained earlier). Hence, $2(a^2b^2 - 2b^2x^2) = a^2b^2 - b^2x^2$, i.e., $3b^2x^2 = a^2b^2$, i.e., $x^2 = \frac{a^2}{3}$, i.e., $x = \frac{a}{\sqrt{3}}$.

So, $y^2 = \frac{a^2b^2 - 2b^2x^2}{a^2} = \frac{a^2b^2 - 2b^2\frac{a^2}{3}}{a^2} = \frac{\frac{1}{3}a^2b^2}{a^2} = \frac{b^2}{3}$, i.e., $y = \frac{b}{\sqrt{3}}$. Now $\frac{z^2}{c^2} = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 1 - \frac{1}{3} - \frac{1}{3}$; or $z = \frac{c}{\sqrt{3}}$. The volume of the ellipsoid $= \frac{4\pi}{3}abc$; volume

of parallelopiped $= \dfrac{8abc}{3\sqrt{3}}$.

The ratio of the volume of the parallelopiped to that of ellipsoid $= \dfrac{8abc}{3\sqrt{3}} : \dfrac{4\pi}{3}abc = 2 : \pi\sqrt{3}$.

Problem 2. Given the sum of the length of the axes of the ellipsoid to find their length such that its volume is maximum.

Solution: The problem is one of maximizing xyz where $x+y+z=s$ is fixed. $xyz = xy(s-x-y) = sxy - x^2y - xy^2$. Let x be first made constant. Then we are to maximize $x\{(s-x)y - y^2\}$. x being constant we are to maximize $(s-x)y - y^2$, By solution to Problem 1 of Chapter I, $y = \dfrac{s-x}{2}$. Likewise keeping y constant and using the solution for y in this we get

$$x = \frac{s-y}{2} = \frac{s - \dfrac{s-x}{2}}{2} = \frac{s+x}{4}, \quad \text{i.e.,} \quad \frac{3x}{4} = \frac{s}{4} \quad \text{or } x = \frac{s}{3}.$$

By symmetry $y = \dfrac{s}{3}, z = \dfrac{s}{3}$. i.e., the axes of the ellipsoid should be equal, i.e., it should be a sphere.

Problem 3. Find x, y such that $x^3 + y^3 - 3axy$ is maximum or minimum for a given $a > 0$.

Solution: Clearly for numerically large x, y $x^3 + y^3 - 3axy$ is large so that we need to consider only its minimization. Keeping x constant we are to minimize $y^3 - 3axy$. By solution to Problem 1, Chapter II, the minimum value of $y^3 - 3axy$ corresponds to $y = \dfrac{\sqrt{3ax}}{\sqrt{3}} = +\sqrt{ax}$. Likewise, keeping y

constant we get $x = +\sqrt{ay}$ for minimizing $y^3 - 3axy$. From these two $x^2 = \left(\sqrt{ay}\right)^2 = ay = a\sqrt{ax}$, i.e., $x^{\frac{3}{2}} = a^{\frac{3}{2}}$ or $x = a$. $y = \sqrt{ax} = \sqrt{a^2} = a$.

Problem 4. To find $x, y > 0$ such that $x^2y^3(a - x - y)$ is maximum, $a > 0$, being given.

Solution: Taking x constant we are to maximize $y^3(a - x) - y^4$. The solution, by Problem 1, Chapter III, the minimum is reached when $\frac{y}{a - x} = \frac{3}{4}$. If y is made constant we are to maximize $x^2(a - x - y)$. i.e., $x^2(a - y) - x^3$. The solution, by Problem 2 of Chapter II, $\frac{x}{a - y} = \frac{2}{3}$,

i.e., $3x = 2(a - y) = 2\left\{a - \frac{3}{4}(a - x)\right\} = \frac{1}{2}\{4a - 3(a - x)\}$,

i.e., $6x = 4a - 3a + 3x$, i.e., $3x = a$,

i.e., $x = \frac{a}{3}$, $y = \frac{3}{4}(a - x) = \frac{3}{4}\left(a - \frac{a}{3}\right) = \left(\frac{3}{4}\right)\left(\frac{2a}{3}\right) = \frac{a}{2}$.

Note: There is an error in the original text where the function considered needs to be $x^3y^2(a - x - y)$ if the solution is to be valid.

Problem 5. To show that the area of a triangle with given perimeter is maximum when it is equilateral.

Solution: If $2p$ is the given perimeter and x, y are two sides, the area $= \sqrt{p(p - x)(p - y)(x + y - p)}$. Thus we are to maximize $(p - x)(p - y)(x + y - p)$. When x is made constant we have to maximize $(p - y)(x + y - p)$, i.e., $py - yx - y^2 + py$

or equivalently $(2p-x)y-y^2$. $y=\dfrac{2p-x}{2}$ by the solution of Problem 1 of Chapter I. Likewise, keeping y constant, we have the solution $x=\dfrac{2p-y}{2}$ and considering both $x=\dfrac{2p-y}{2}=\dfrac{2p-\dfrac{2p-x}{2}}{2}=\dfrac{2p+x}{4}$ i.e., $3x=2p$, i.e., $x=\dfrac{2p}{3}$. So $y=\dfrac{2p}{3}$. Thus, the triangle is equilateral.

Problem 6. To find the rectangular parallelopiped of given surface area which has maximum volume.

Solution If x,y,z are the length of the edges, we are to maximize xyz under the condition $xy+yz+zx=a$, a constant. Now, $z=\dfrac{a-xy}{y+x}$ so that $xyz=\dfrac{xy(a-xy)}{y+x}$. When x is kept constant we are to maximize $\dfrac{y(a-xy)}{y+x}$. Let r be the maximum. Then $y(a-xy)=r(y+x)$, i.e., $xy^2+y(r-a)+rx=0$, i.e., $y^2-y\dfrac{(a-r)}{x}+r=0$ i.e., $r=\left(\dfrac{a-r}{2x}\right)^2-\left(y-\dfrac{a-r}{2x}\right)^2$ so that r is maximum when $y=\dfrac{a-r}{2x}$ and when r is maximum $r=\left(\dfrac{a-r}{2x}\right)^2$, i.e., $r^2-2ar+a^2=4x^2r$, i.e., $r^2-2(a+2x^2)r=-a^2$, i.e., $\left(r-(a+2x^2)\right)^2=\left(a+2x^2\right)^2-a^2.=4ax^2+4x^4=4x^2(a+x^2)$.

So, $r=a+2x^2\pm 2x\sqrt{a+x^2}$. Hence $y=\dfrac{a-r}{2x}=-x\pm\sqrt{a+x^2}$. y needs to be positive. So $y=-x+\sqrt{a^2+x^2}$. Keeping y constant, we have the solution $x=-y+\sqrt{a+y^2}$.

Hence $x+y=\sqrt{a+x^2}$, i.e., $x^2+2xy+y^2=a+x^2$,

i.e., $2xy = a - y^2$ or $x = \dfrac{a - y^2}{2y} = -y + \sqrt{a + y^2}$,

i.e., $\dfrac{a + y^2}{2y} = \sqrt{a + y^2}$.

i.e., $(a + y^2)^2 = 4y^2(a + y^2)$, y^2 being positive

i.e., $a + y^2 = 4y^2$, i.e., $3y^2 = a$ or $y = \sqrt{\dfrac{a}{3}}$.

By symmetry $x = z = \sqrt{\dfrac{a}{3}}$. i.e., the required parallelopiped is a cube.

Problem 7. To inscribe the triangle of greatest area in a given circle.

Solution: If the triangle is ABC, R the given circumradius,

$$\begin{aligned}\Delta \equiv \Delta ABC &= \frac{abc}{4R} = \frac{2R\sin A.2R\sin B.2R\sin(180^\circ - (A + B))}{4R} \\ &= 2R^2 \sin A \sin B \sin(A + B) \\ &= 2R^2 \sin A \sin B(\sin A \cos B + \cos A \sin B).\end{aligned}$$

Putting $\sin A = x$, $\sin B = y$,

$$\Delta = 2R^2 xy(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}).$$

We are to maximize $xy(x\sqrt{1 - y^2} + y\sqrt{1 - x^2})$. Making x constant, we are to maximize $yx\sqrt{1 - y^2} + y^2\sqrt{1 - x^2}$; or, equivalently, $y\sqrt{1 - y^2} + \dfrac{\sqrt{1 - x^2}}{x}y^2$. If the maximum is r, and we write

$$X = \frac{\sqrt{1 - x^2}}{x}$$

for convenience, $y^2 - y^4 = r^2 - 2Xy^2r + X^2y^4$

i.e., $$y^4(X^2+1) - (2Xr+1)y^2 = -r^2.$$

i.e., $$y^4 - \frac{2Xr+1}{X^2+1}y^2 = -\frac{r^2}{X^2+1}$$

i.e., $$\frac{r^2}{X^2+1} = \frac{1}{4}\left(\frac{2Xr+1}{X^2+1}\right)^2 - \left(y^2 - \frac{1}{2}\frac{2Xr+1}{X^2+1}\right)^2.$$

When r is maximum $y^2 = \frac{1}{2}\left(\frac{2Xr+1}{X^2+1}\right)$. Also, the greatest value of r is given by $\frac{r^2}{X^2+1} = \frac{1}{4}\left(\frac{2Xr+1}{X^2+1}\right)^2$, i.e., $4(X^2+1)r^2 = (2Xr+1)^2 = 4X^2r^2 + 4Xr + 1$, i.e., $4r^2 - 4Xr - 1 = 0$, i.e., $r^2 - Xr - \frac{1}{4} = 0$ so that

$$r = \frac{X + \sqrt{X^2+1}}{2},\ y^2 = \frac{2Xr+1}{2(X^2+1)} = \frac{X^2 + X\sqrt{X^2+1}+1}{2(X^2+1)}.$$

Putting $X^2 = \frac{1-x^2}{x^2}$, $y^2 = \frac{1+\sqrt{1-x^2}}{2}$,

i.e., $\sin^2 B = \frac{1+\cos A}{2}$ or $\cos A = 2\sin^2 B - 1 = \cos 2B$.

Likewise, $\sin^2 A = \frac{1+\cos B}{2}$ when y is treated as constant.

So

$$\cos^2 A = 1 - \sin^2 A = 1 - \frac{1+\cos B}{2} = \frac{1-\cos B}{2}.$$

Then $\cos B = 1 - 2\cos^2 A = 1 - 2\cos^2 2B = -(2\cos^2 2B - 1) = -\cos 4B$. Hence $4B = (2n+1)\pi + B$ for suitable integer n. $0 < B < \pi$ so that $4B = \pi + B$ or $3B = \pi$ or $B = 60°$. By symmetry $A = 60°$, i.e., ΔABC is equilateral.

Problem 8. To find a point G within a triangle ΔABC such that $AG^2 + BG^2 + CG^2$ is minimum.

Solution: Let D be the foot of the altitude from A and let E and F be the feet of the perpendiculars from G to AD and BC respectively (See Fig 4.1)

Write $BD = c$, $BC = b$, $AD = a$ (constants). Let $BF = x$, $FG = y$ (variables) where G is the point required.

Then

$$\begin{aligned} AG^2 &= AE^2 + EG^2 \\ &= AE^2 + (BF - BD)^2, \\ BG^2 &= BF^2 + FG^2, \\ CG^2 &= GF^2 + CF^2 \\ &= GF^2 + (BC - BF)^2. \end{aligned}$$

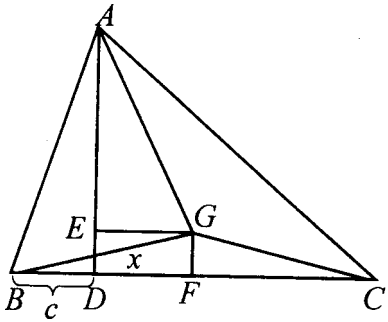


Fig. 4.1

$$\begin{aligned} \text{i.e., } AG^2 &= (a-y)^2 + (x-c)^2, \\ BG^2 &= x^2 + y^2, \\ CG^2 &= (b-x)^2 + y^2. \end{aligned}$$

Hence

$AG^2 + BG^2 + CG^2 = (a-y)^2 + (x-c)^2 + x^2 + y^2 + (b-x)^2 + y^2$.

We are to minimize therefore $3x^2 + 3y^2 - 2(b+c)x - 2ay$. Treating x constant we are to minimize $3y^2 -$

$2ay = r$. or $y^2 - \frac{2a}{3}y = \frac{r}{3}$, i.e., $\left(y - \frac{a}{3}\right)^2 = \frac{a^2}{9} + \frac{r}{3}$ or $\frac{r}{3} = \left(y - \frac{a}{3}\right)^2 - \frac{a^2}{9}$ which is minimum when $y = \frac{a}{3}$. Likewise considering y constant we have to minimize $3x^2 - 2(b+c)x$ for which we have $x = \frac{b+c}{3}$ as the solution.

Actually, $AG^2 + BG^2 + CG^2$

$$
\begin{aligned}
&= 3x^2 + 3y^2 - 2(b+c)x - 2ay + a^2 + b^2 + c^2 \\
&= 3x^2 - 2(b+c)x + b^2 + c^2 + 3y^2 - 2ay + a^2 \\
&= 3\left\{\left(x - \frac{b+c}{3}\right)^2 - \frac{(b+c)^2}{9}\right\} \\
&\quad +3\left\{\left(y - \frac{a}{3}\right)^2 - \frac{a^2}{9}\right\} + (b^2 + c^2) + a^2 \\
&= 3\left(x - \frac{b+c}{3}\right)^2 + \left(b^2 + c^2\right) - \frac{(b+c)^2}{3} + a^2 - \frac{a^2}{3} \\
&= b^2 + c^2 - \frac{(b+c)^2}{3} + a^2 - \frac{a^2}{3}
\end{aligned}
$$

when $x = \frac{b+c}{3}$. Note that

$$
\begin{aligned}
b^2 + c^2 - \frac{(b+c)^2}{3} &= \frac{2b^2 + 2c^2 - 2bc}{3} \\
&= \frac{2}{3}(b^2 + c^2 - bc) > \frac{2}{3}(b^2 + c^2 - 2bc) \\
&= \frac{2}{3}(b-c)^2 > 0
\end{aligned}
$$

and $\quad a^2 - \frac{a^2}{3} = \frac{2a^2}{3} > 0.$

Problem 9. To find a point within a tetrahedron such that the sum of the squares of distances of this point from the vertices of the tetrahedron is the least possible.

Solution: Let ABC be the base triangle of the tetrahedron and E its vertex (See Fig 4.2). Let EG be the altitude from E to the base ABC and let $EG = a$. Let K be the point required and KH be the perpendicular from K to the base ABC with its foot at H. Let HD be perpendicular to AC meeting AC at D in the plane ABC. Let $HD = y$, $AD = x$ (variables). Let GF be perpendicular to AC in the plane ABC meeting AC at F. Let $GF = b$, $AF = c$ (constants). Let HN be perpendicular to GF meeting it at N, again, in the plane ABC. Then $HN = DF = x - c$ and $GN = GF - HD = b - y$. Let $AC = d$ (constant). Then $CD = AC - AD = d - x$. Then $KL = GH$. Join K and B, B and H. By construction, $KH \perp HB$. Let the length of the altitude of ΔABC from B be of length e (a constant). As in the solution for Problem 8 (See Fig 4.3), we can show that $HB^2 = (f - x)^2 + (e - y)^2$ when $AM = f$, M being the foot of the perpendicular from B to AC.

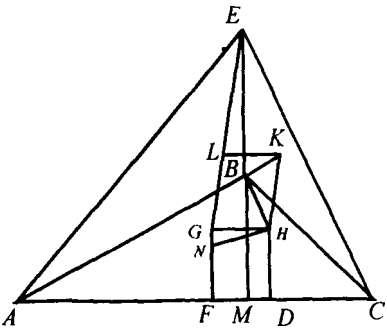


Fig. 4.2

Let $KH = z$, a variable. Then $KB^2 = (f-x)^2 + (e-y)^2 + z^2$. Now,

$$\begin{aligned} AK^2 &= AH^2 + HK^2 \\ &= AD^2 + HD^2 + HK^2 \\ &= x^2 + y^2 + z^2 \end{aligned} \tag{1}$$

$$BK^2 = (f-x)^2 + (e-y)^2 + z^2 \text{ (as proved earlier)} \tag{2}$$

$$\begin{aligned} CK^2 &= CH^2 + HK^2 = CD^2 + HD^2 + HK^2 \\ &= (d-x)^2 + y^2 + z^2 \end{aligned} \tag{3}$$

$$\begin{aligned} EK^2 &= KL^2 + LE^2 = KL^2 + (EG - KH)^2 \\ &= GH^2 + (EG - KH)^2 \\ &= FD^2 + GN^2 + (EG - KH)^2 \\ &= (x-c)^2 + (b-y)^2 + (a-z)^2 \quad \text{(see Fig 4.4.)} \end{aligned} \tag{4}$$

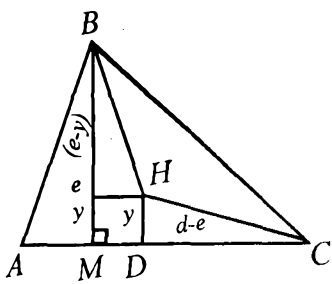


Fig. 4.3

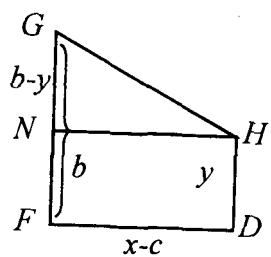


Fig. 4.4

From (1)-(4)

$$\begin{aligned} AK^2 + BK^2 + CK^2 + EK^2 \\ = \; & x^2 + y^2 + z^2 + (f-x)^2 + (e-y)^2 + z^2 \\ & + (d-x)^2 + y^2 + z^2 \\ & + (x-c)^2 + (b-y)^2 + (a-z)^2 \\ = \; & 4(x^2+y^2+z^2) - 2x(f+d+c) \\ & - 2y(e+b) - 2az + f^2 + e^2 + d^2 + c^2 + b^2 + a^2 \\ = \; & 4(x^2+y^2+z^2) - 2x(f+d+c) - 2y(e+b) \\ & - 2az + a^2 + b^2 + c^2 + d^2 + e^2 + f^2. \end{aligned}$$

Making x, y, z respectively constant, we are to minimize

$$4(y^2+z^2) - 2y(e+b) - 2az$$

$$4(x^2+z^2) - 2x(f+d+c) - 2az$$

$$4(x^2+y^2) - 2x(f+d+c) - 2y(e+b).$$

Omitting constant multiples, the maximum is reached in each case when

$$\begin{aligned} (y^2+z^2) - y\frac{(e+b)}{2} - \frac{az}{2} \; = \; & \left(y - \frac{e+b}{4}\right)^2 \\ & + \left(z - \frac{a}{4}\right)^2 - \frac{(e+b)^2}{16} - \frac{a^2}{16}, \quad \text{etc.} \end{aligned}$$

are minimum. This happens when

$$y = \frac{e+b}{4}, \; z = \frac{a}{4}, \; x = \frac{f+d+c}{4}.$$

Note that $4(x^2+y^2+z^2)-2x(f+d+c)-2y(e+b)-2az$

$$+a^2+b^2+c^2+d^2+e^2+f^2$$

$$= \left(2x-\frac{f+d+c}{2}\right)^2+\left(2y-\frac{e+b}{2}\right)^2+\left(2z-\frac{a}{2}\right)^2+a^2+b^2$$

$$+c^2+d^2+e^2+f^2-\left(\frac{f+d+c}{2}\right)^2-\left(\frac{e+b}{2}\right)^2-\left(\frac{a}{2}\right)^2$$

$$\geq 0 \quad \left(\text{since } a^2+b^2+c^2+d^2+e^2+f^2-\left(\frac{f+d+c}{2}\right)^2\right.$$

$$\left.-\left(\frac{e+b}{2}\right)^2-\left(\frac{a}{2}\right)^2\right.$$

$$= \frac{1}{4}\Big[3a^2+4(e^2+b^2)-(e+b)^2+4(f^2+d^2+c^2)$$

$$(f+d+c)^2\Big] \geq 0).$$

Remark: There seems to be an error in the solution in the text in which AC and AM are both supposed to be equal to d.

Problem 10. To find $x, y > 0$ such that $(x+1)(y+1)(z+1)$ is maximum z being related to x, y by $a^x b^y c^z = A$, $a, b, c, A > 0$ being constants.

Solution: Taking logarithms the constraint becomes $px + my + nz = q$, where $p = \log a$, $m = \log b$, $n = \log c$ and $q = \log A$. Instead of $(x+1)(y+1)(z+1)$, we can consider $(px+p)(my+m)(nz+n) = pmn(x+1)(y+1)(z+1)$.

$$px+p+my+n+nz+n = px+my+nz+p+m+n = q+p+m+n.$$

Thus, the problem reduces to maximizing XYZ where $X = px+p$, $Y = mx+m$, $Z = nz+n$ under the constraint

$X+Y+Z=s=q+p+m+n$. The solution is as in Problem 2, viz., $X=Y=Z=\frac{s}{3}$, i.e.,

$$x=\frac{q-2p+m+n}{3p},\ y=\frac{q+p-2m+n}{3m},\ z=\frac{q+p+m-2n}{3n}.$$

The maximum value of $(x+1)(y+1)(z+1)$ is

$$\frac{\log(Aabc)}{27\log a\log b\log c}.$$

Problem 11. To inscribe a triangle of maximum perimeter in a given circle.

Solution: Let R be the radius of the circle. If it circumscribes the ΔABC, then, by the sine formula (with the usual notation)

$$\begin{aligned} a+b+c &= 2R(\sin A+\sin B+\sin C)\\ &= 2R(\sin A+\sin B+\sin(180-\overline{A+B}))\\ &= 2R(\sin A+\sin B+\sin(A+B)). \end{aligned}$$

Thus, we are to maximize the function $\sin A+\sin B+\sin(A+B)$ where A, B are variables. $(\sin A+\sin B+\sin(A+B)=\sin A+\sin B+\sin A\cos B+\cos A\sin B=\sin A(1+\cos B)+\sin B(1+\cos A))$. Put $\sin A=x$, $\sin B=y$. Then keeping x constant we are to maximize

$$x\sqrt{1-y^2}+y\left(1+\sqrt{1-x^2}\right).$$

For convenience, write $x=l$, $1+\sqrt{1-x^2}=m$ (presently constant). Then we are to maximize $l\sqrt{1-y^2}+my=r$ say. Squaring and rearranging

$$l^2(1-y^2)=(r-my)^2$$

i.e., $$l^2 - (l^2 + m^2)y^2 + 2rmy = r^2$$

i.e., $$\frac{l^2}{l^2 + m^2} - \left[y^2 - 2\frac{rm}{l^2 + m^2}y\right] = \frac{r^2}{l^2 + m^2}$$

i.e., $$\frac{l^2}{l^2 + m^2} + \frac{r^2m^2}{(l^2 + m^2)^2} - \left\{y - \frac{rm}{l^2 + m^2}\right\}^2 = \frac{r^2}{l^2 + m^2}$$

i.e., $$\frac{l^2}{l^2 + m^2} - \left\{y - \frac{rm}{l^2 + m^2}\right\}^2 = r^2\left[\frac{1}{l^2 + m^2} - \frac{m^2}{(l^2 + m^2)^2}\right]$$

i.e., $$\frac{l^2}{l^2 + m^2} - \left\{y - \frac{rm}{l^2 + m^2}\right\}^2 = r^2\frac{l^2}{(l^2 + m^2)^2}$$

r is maximum when $y = \dfrac{rm}{l^2 + m^2}$. Then

$$r^2\frac{l^2}{(l^2 + m^2)^2} = \frac{l^2}{l^2 + m^2}$$

i.e., $r^2 = l^2 + m^2$, i.e.,

$$\begin{aligned} y &= \frac{m\sqrt{l^2 + m^2}}{l^2 + m^2} = \frac{m}{\sqrt{l^2 + m^2}} \\ &= \frac{1 + \sqrt{1 - x^2}}{\sqrt{x^2 + 1 + 1 - x^2 + 2\sqrt{1 - x^2}}} \\ &= \frac{1 + \sqrt{1 - x^2}}{\sqrt{2}\sqrt{1 + \sqrt{1 - x^2}}} \\ &= \frac{\sqrt{1 + \sqrt{1 - x^2}}}{\sqrt{2}}. \end{aligned}$$

Likewise, making y constant we have maximum when

$$x = \frac{\sqrt{1 + \sqrt{1 - y^2}}}{\sqrt{2}}.$$

$2y^2 = 1 + \sqrt{1 - x^2}$. So, $(2y^2 - 1)^2 = 1 - x^2$, i.e., $4y^4 - 4y^2 + 1 = 1 - x^2$ or $x^2 = 4y^2 - 4y^4$. Likewise $y^2 = 4x^2 - 4x^4$.

Thus. $y^2 = 4x^2(1-x^2) = (16y^2 - 16y^4)(2y^2 - 1)^2$

$$= -64y^8 + 128y^6 - 80y^4 + 16y^2 .$$

Hence $$y^6 - 2y^4 + \frac{5}{4}y^2 - \frac{15}{64} = 0.$$

Putting $y^2 = z$, we have $z^3 - 2z^2 + \frac{5}{4}z - \frac{15}{64} = 0$ which, on inspection has $z = \frac{3}{4}$ as a root, i.e., $y^2 = \frac{3}{4}$ or $y = \frac{\sqrt{3}}{2}$ so that $\sin B = \frac{\sqrt{3}}{2}$; likewise $\sin A = \frac{\sqrt{3}}{2}$. Hence $A = B = C = 60°$

Problem 12. To find $x, y, z > 0$ such that $\frac{xyz}{(x+a)(x+y)(y+z)(z+c)}$ is maximum where a, c are positive constants.

Solution: Making y and z constant, we are to maximize $\frac{x}{(x+a)(x+y)}$, or, equivalently, to minimize $\frac{(x+a)(x+y)}{x} = r$, say. Then $x^2 + (a+y)x + ay = rx$. i.e., $x^2 + (a+y-r)x + ay = 0$, i.e., $\left(x + \frac{a+y-r}{2}\right)^2 + ay = \frac{(a+y-r)^2}{4}$. r is minimum when $x + \frac{a+y-r}{2} = 0$, i.e., $r = 2x + a + y$ and for this value $\frac{(a+y-r)^2}{4} = ay$. i.e., $(a+y-r)^2 = 4ay$, i.e., $4x^2 = 4ay$, i.e., $x = \sqrt{ay}$. Likewise $y = \sqrt{xz}$ when x and z are constant and $z = \sqrt{yc}$ when x, y are made constant. So $x^2 = ay = a\sqrt{xz}$, i.e., $x^4 = a^2xz$. i.e., $x^3 = a^2z$. But $z^2 = yc = c\sqrt{xz}$. i.e., $z^4 = c^2xz$, i.e., $z^3 = c^2x$. So, $c^6x^3 = z^9 = c^6a^2z$, i.e., $z^8 = c^6a^2$, i.e., $z = c^{\frac{3}{4}}a^{\frac{1}{4}}$, $z = \sqrt[4]{c^3a}$. Similarly, $x = \sqrt[4]{a^3c}$. Now, $y = \sqrt{xz} = \sqrt{a^{\frac{3}{4}}.c^{\frac{1}{4}}.c^{\frac{3}{4}}a^{\frac{1}{4}}} = \sqrt{ac}$.

Problem 13. To find the rectangular parallelopiped of minimum surface area with given volume.

Solution: The problem reduces to minimizing $xy + yz + zx$ under the constraint $xyz = a$. Thus, we are to minimize $xy + \frac{a}{x} + \frac{a}{y} = \frac{x^2y^2 + ay + ax}{xy}$. When y is constant we have to minimize $\frac{y^2}{x}\left[x^2 + \frac{ax}{y^2} + \frac{a}{y}\right]$.

Equivalently, $\frac{1}{x}\left[x^2 + \frac{ax}{y^2} + \frac{a}{y}\right] = r$,

i.e., $x^2 + \frac{ax}{y^2} + \frac{a}{y} = rx$, i.e., $x^2 + \left(\frac{a}{y^2} - r\right)x + \frac{a}{y} = 0$,

i.e., $\left[x + \frac{1}{2}\left(\frac{a}{y^2} - r\right)\right]^2 + \frac{a}{y} = \frac{1}{4}\left(\frac{a}{y^2} - r\right)^2$.

r is minimum when $x = \frac{1}{2}\left(r - \frac{a}{y^2}\right)$ and the minimum value of r is given by $\frac{1}{4}\left(\frac{a}{y^2} - r\right)^2 = \frac{a}{y}$, i.e., $(a - ry^2)^2 = 4ay^3$;

i.e., $ry^2 - a = 2\sqrt{ay^3}$, i.e., $r = \frac{(a + 2\sqrt{ay^3})}{y^2} = \frac{a}{y^2} + 2\sqrt{\frac{a}{y}}$.

So $x = \frac{2\sqrt{\frac{a}{y}}}{2} = \sqrt{\frac{a}{y}}$. Likewise $y = \sqrt{\frac{a}{x}}$. $z = \frac{a}{xy} = \frac{a}{\frac{a}{\sqrt{xy}}} = \sqrt{xy}$, i.e., $z^2 = xy$.

Hence $x^2 = \frac{a}{y}$ and $y^2 = \frac{a}{x}$. So $y^4 = \frac{a^2}{x^2}$ or $x^2 = \frac{a^2}{y^4}$. Therefore, $\frac{a}{y} = x^2 = \frac{a^2}{y^4}$ or $1 = \frac{a}{y^3}$ or $y = \sqrt[3]{a}$. Similarly $x = \sqrt[3]{a}$. So $z = \sqrt{xy} = \sqrt{a^{2/3}} = a^{1/3}$. Thus the parallelopiped is a cube.

Note: In the solution we take $x = \frac{1}{2}\left(r - \frac{a}{y^2}\right)$ since $\frac{a}{y^2} = \frac{xyz}{y^2} = \frac{xz}{y} < xz + yz + zx = r\,\frac{a}{y^2} - r < 0$.

Problem 14. To find a point P inside a given convex quadrilateral $ABCD$ such that $PA^2 + PB^2 + PC^2 + PD^2$ is minimum.

Solution: Let $AD = b$, $AB = a$, $BC = c$. (See Fig 4.5). Let F, N, G be the feet of the perpendiculars from D, P, C respectively on AB. Let E, P, H with E, H lying on DF and CG be parallel to AB. Let $AN = x$, $NP = y$ (variables). Now $FD = b\sin A$, $FA = b\cos A$, $GC = c\sin B$, $BG = c\cos B$

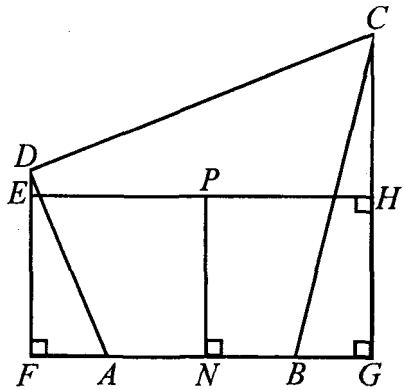


Fig. 4.5

$EP = FN = FA + AN = x + b\cos A;$

$ED = DF - EF = DF - PN = b\sin A - y.$

$PH = NG = NB + BG = a - x + c\cos B,$

$CH = GC - HG = GC - PN = c\sin B - y.$

So,
$$AP^2 = x^2 + y^2 \quad (1)$$
$$BP^2 = (a-x)^2 + y^2 \quad (2)$$
$$CP^2 = (a - x + c\cos B)^2 + (c\sin B - y)^2 \quad (3)$$
$$DP^2 = (x + b\cos A)^2 + (b\sin A - y)^2 \quad (4)$$

Hence

$$AP^2+BP^2+CP^2+DP^2 = 2y^2+x^2+(a-x)^2+(a-x+c\cos B)^2 + (c\sin B - y)^2 + (x + b\cos A)^2 + (b\sin A - y)^2$$

is to be minimized. First making y constant, we are to minimize

$$x^2 + (a-x)^2 + (a - x + c\cos B)^2 + (x + b\cos A)^2$$
$$= 4x^2 - 2(2a - b\cos A + c\cos B)x + 2a^2 + b^2\cos^2 A + c^2\cos^2 B + 2ac\cos B.$$

This quadratic attains its minimum, when $x = \dfrac{2a - b\cos A + c\cos B}{4}$. When x is made constant we obtain

$$2y^2 + (c\sin B - y)^2 + (b\sin A - y)^2$$
$$= 4y^2 - 2(b\sin A + c\sin B)y + b^2\sin^2 A + c^2\sin^2 B.$$

so that the minimum is reached when $y = \dfrac{b\sin A + c\sin B}{4}$

Note: There is a misprint in the original text where $\cos B$ appears instead of $\sin B$ in the value of y.

Problem 15. Find x, y, z such that $x^2 + y^2 + z^2 = 1$ and $ax + by + cz$ is a maximum.

Solution: We are to maximize $ax + by + c\sqrt{1 - x^2 - y^2}$. First, letting y to be constant we are to maximize $ax + c\sqrt{1 - x^2 - y^2} = a\left[x + \dfrac{c}{a}\sqrt{1 - x^2 - y^2}\right]$ omitting the factor a, let $x + \dfrac{c}{a}\sqrt{1 - x^2 - y^2} = r$.

Then $$(r - x)^2 = \frac{c^2}{a^2}(1 - x^2 - y^2)$$

i.e., $$r^2 - 2rx + x^2 = \frac{c^2}{a^2}(1 - x^2 - y^2)$$

i.e., $$x^2\left(1 + \frac{c^2}{a^2}\right) - 2rx + r^2 = \frac{c^2}{a^2}(1 - y^2)$$

i.e., $$x^2(c^2 + a^2) - 2ra^2x + r^2a^2 = c^2(1 - y^2)$$

i.e., $$x^2 - \frac{2ra^2x}{c^2+a^2} + \frac{r^2a^2}{c^2+a^2} = \frac{c^2}{c^2+a^2}(1-y^2)$$

i.e., $$\frac{r^2a^2}{c^2+a^2} - \frac{r^2a^4}{(c^2+a^2)^2} = \frac{c^2}{c^2+a^2}(1-y^2) - \left(x - \frac{ra^2}{c^2+a^2}\right)^2$$

i.e., $$\frac{r^2a^2}{(c^2+a^2)^2}[c^2+a^2-a^2] = \frac{c^2}{c^2+a^2}(1-y^2) - \left(x - \frac{ra^2}{c^2+a^2}\right)^2.$$

Thus r^2 and so r is maximum when $x = \dfrac{ra^2}{c^2+a^2}$ and the maximum is given by $r^2 = \dfrac{(c^2+a^2)}{a^2}(1-y^2)$. Thus

$$x^2 = \frac{r^2a^4}{(c^2+a^2)^2} = \frac{a^2}{c^2+a^2}(1-y^2)$$

i.e., $x = a\sqrt{\dfrac{1-y^2}{c^2+a^2}}$. Likewise, letting x to be constant,

we get $y = b\sqrt{\dfrac{1-x^2}{c^2+b^2}}$. Now, $x^2 = a^2\dfrac{(1-y^2)}{c^2+a^2}$.

So, $y^2 = \dfrac{a^2-(c^2+a^2)x^2}{a^2}$.

i.e., $y^2 = 1 - \dfrac{c^2+a^2}{a^2}x^2 = \dfrac{b^2-b^2x^2}{c^2+b^2}$.

So, $$\left[\frac{c^2+a^2}{a^2} - \frac{b^2}{c^2+b^2}\right]x^2 = 1 - \frac{b^2}{c^2+b^2} = \frac{c^2}{c^2+b^2}.$$

i.e., $$\frac{a^2b^2+a^2c^2+b^2c^2+c^4-a^2b^2}{a^2(b^2+c^2)}x^2 = \frac{c^2}{c^2+b^2}$$

i.e., $$\frac{a^2b^2+a^2c^2+b^2c^2+c^4-a^2b^2}{a^2}x^2 = c^2$$

i.e., $$(a^2b^2+a^2c^2+b^2c^2+c^4-a^2b^2)x^2 = c^2a^2$$

i.e., $$c^2(a^2+b^2+c^2)x^2 = c^2a^2$$

i.e., $$x^2 = \frac{a^2}{a^2+b^2+c^2}.$$

So $x = \dfrac{a}{\sqrt{a^2+b^2+c^2}}$ and likewise $y = \dfrac{b}{\sqrt{a^2+b^2+c^2}}$.

Then $z = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

Problem 16. To find a point P within ΔABC such that $PA^2 + PB^2 + PC^2$ is minimum.

Note: This is Problem 8 of this Chapter for which a different solution is now given.

Solution: As in Fig 4.6, let AD be the altitude from A and let PN be the perpendicular from the desired point on BC. Let $CN = x$, $NP = y$ (variables) and let us adhere to the usual notation relating to a triangle. Then $AD = b\sin C$, $CD = b\cos C$ and

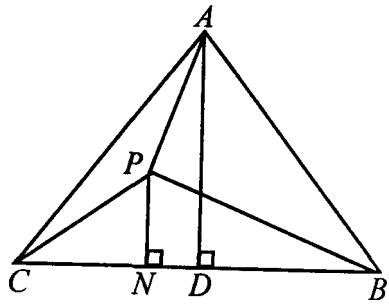


Fig. 4.6

$$CP^2 = x^2 + y^2; \quad BP^2 = y^2 + (a-x)^2 = y^2 + x^2 + a^2 - 2ax,$$

$$AP^2 = (b\cos C - x)^2 + (b\sin C - y)^2 = b^2 + x^2 + y^2 - 2b(x\cos C + y\sin C).$$

We are to minimize

$$AP^2 + BP^2 + CP^2 = 3x^2 + 3y^2 + a^2 + b^2 - 2ax - 2b(x\cos C + y\sin C).$$

Making y constant, we are to minimize

$$x^2 - \frac{2a}{3}x - \frac{2b}{3}x\cos C = x^2 - \frac{2x}{3}(a + b\cos C) = r, \quad \text{say.}$$

Then $r = -\left(\frac{a + b\cos C}{3}\right)^2 + \left(x - \frac{a + b\cos C}{3}\right)^2$ so that the minimum is attained when $x = \frac{a + b\cos C}{3}$. Making x constant we are to minimize $3y^2 - 2by\sin C = r$ so that $\left(y - \frac{b\sin C}{3}\right)^2 - \frac{b^2\sin^2 C}{3^2} = r$ is minimum when $y = \frac{b\sin C}{3}$.

Problem 17. To find a point P in the interior of ΔABC such that $PD^2 + PE^2 + PF^2$ is minimum where D, E, F are the feet of the perpendiculars from P on BC, CA, AB respectively. (See Fig 4.7).

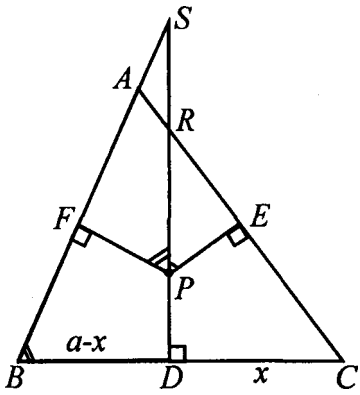


Fig. 4.7

Solution: Let $CD = x, PD = y$ (variables). Let PD meet AC at R and BA produced at S. Now,

$$\begin{aligned} PE^2 &= RP^2\cos^2 R\hat{P}E = RP^2\cos^2 C \\ &\qquad \text{(using property of a cyclic quadrilateral)} \\ &= (RD - PD)^2\cos^2 C = (x\tan C - y)^2\cos^2 C \\ &= (x\sin C - y\cos C)^2 \\ PF^2 &= PS^2\cos^2 B = (SD - PD)\cos^2 B \\ &= ((a - x)\tan B - y)^2\cos^2 B \\ &= ((a - x)\sin B - y\cos B)^2. \end{aligned}$$

Thus $PD^2 + PE^2 + PF^2$

$= y^2 + (x\sin C - y\cos C)^2 + \{(a - x)\sin B - y\cos B\}^2$.

We are thus to minimize

$$y^2(1+\cos^2 C+\cos^2 B)+x^2(\sin^2 C+\sin^2 B)$$

$$-2xy(\sin C\cos C-\sin B\cos B)$$

$$-2ax\sin^2 B-2ay\sin B\cos B+a^2\sin^2 B.$$

If x is constant we are to consider

$$y^2(1+\cos^2 C+\cos^2 B)-2xy(\sin C\cos C-\sin B\cos B)$$

$$-2ay\sin B\cos B$$

i.e., $y^2(1+\cos^2 C+\cos^2 B)-$

$$2y\{x(\sin C\cos C-\sin B\cos B)+a\sin B\cos B\}$$

The maximum is reached when

$$y=\frac{(\sin C\cos C-\sin B\cos B)+a\sin B\cos B}{(1+\cos^2 C+\cos^2 B)}.$$

When y is constant we are to consider

$$x^2(\sin^2 C+\sin^2 B)-2xy(\sin C\cos C-\sin B\cos B)-2ax\sin^2 B.$$

The minimum is reached when

$$x=\frac{y(\sin C\cos C-\sin B\cos B)+a\sin^2 B}{\sin^2 C+\sin^2 B}.$$

For convenience, let

$$\begin{aligned}\cos C\sin C-\cos B.\sin B &= P\\ a\cos B\sin B &= S\\ 1+\cos^2 C+\cos^2 B &= T\\ a\sin^2 B &= Q\\ \sin^2 B+\sin^2 C &= R\end{aligned}$$

Then $x = \dfrac{Py+Q}{R}$, $y = \dfrac{Px+S}{T}$. So, $x = \dfrac{Ty-S}{P}$,

i.e., $\dfrac{Py+Q}{R} = \dfrac{Ty-S}{P}$, i.e., $P^2y + PQ = RTy - RS$,

i.e., $y(P^2 - RT) = -RS - PQ$ i.e., $y = \dfrac{RS+PQ}{RT-P^2}$.

This simplifies to

$$\begin{aligned} y &= \frac{a\sin A\sin B\sin C}{2(1-\cos^2 B + \cos B\cos C\sin B\sin C)} \\ &= \frac{abc\sin A}{a^2+b^2+c^2} \end{aligned}$$

using the sine and cosine formulas.

Also

$$\begin{aligned} x &= \frac{y(\sin C\cos C - \sin B\cos B) + a\sin^2 B}{\sin^2 C + \sin^2 B} \\ &= \frac{\frac{abc\sin A}{a^2+b^2+c^2}(\sin C\cos C - \sin B\cos B) + a\sin^2 B}{\sin^2 C + \sin^2 B} \\ &= \frac{\frac{abc}{a^2+b^2+c^2}\frac{a}{2R}(\frac{c}{2R}\cos C - \frac{b}{2R}\cos B) + a\frac{b^2}{(2R)^2}}{(\frac{C}{2R})^2 + (\frac{b}{2R})^2} \\ &= \frac{\frac{a^2bc}{a^2+b^2+c^2}(c\cos C - b\cos B) + ab^2}{c^2+b^2} \\ &= \frac{a^2bc(c\dfrac{a^2+b^2-c^2}{2ab} - b\dfrac{a^2+c^2-b^2}{2ac}) + ab^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)(b^2+c^2)} \\ &= \frac{ac^2(a^2+b^2-c^2) - ab^2(a^2+c^2-b^2) + 2ab^2(a^2+b^2+c^2)}{2(a^2+b^2+c^2)(b^2+c^2)} \end{aligned}$$

$$= \frac{a[a^2c^2 - c^4 - a^2b^2 + b^4 + 2a^2b^2 + 2b^4 + 2b^2c^2]}{2(a^2+b^2+c^2)(b^2+c^2)}$$

$$= \frac{a[(b^4 - c^4) + a^2b^2 + a^2c^2 + 2b^4 + 2b^2c^2]}{2(a^2+b^2+c^2)(b^2+c^2)}$$

$$= \frac{a[(b^2 - c^2)(b^2+c^2) + a^2(b^2+c^2) + 2b^2(b^2+c^2)]}{2(a^2+b^2+c^2)(b^2+c^2)}$$

$$= \frac{a(b^2 - c^2 + a^2 + 2b^2)}{2(a^2+b^2+c^2)}$$

$$= \frac{a(a^2+b^2+c^2) + 2(b^2 - c^2)}{2(a^2+b^2+c^2)}$$

$$= \frac{a}{2} + \frac{a(b^2 - c^2)}{a^2+b^2+c^2}$$

Problem 18. Find $x, y, z > 0$ such that $ax^2y^3z^4 - x^3y^3z^4 - x^2y^4z^4 - x^2y^3z^5$. is maximum.

Solution: When x, y are treated as constants and z is variable, we have to maximize

$$az^4 - xz^4 - yz^4 - z^5 = (a - x - y)z^4 - z^5$$

for which, according to Problem 10 of Chapter III, we have the solution

$$z = \frac{4(a - x - y)}{5} \text{ or } 4x + 4y + 5z = 4a. \qquad (1)$$

Letting x, z constant we have to maximize

$$ay^3 - xy^3 - y^4 - zy^3 = (a - x - z)y^3 - y^4$$

for which the solution is

$$y = \frac{3(a - x - z)}{4} \quad \text{or} \quad 3x + 4y + 3z = 3a \qquad (2)$$

When y, z are made constant, we are to consider $(a - y - z)x^2 - x^3$ and according to Problem 2 of Chapter II,

$$x = \frac{2}{3}(a - y - z) \quad \text{or} \quad 3x + 2y + 2z = 2a \tag{3}$$

From (3) and (2), by subtraction

$$2y + z = a \tag{4}$$

(1) × 3 − (3) × 4 gives

$$4y + 7z = 4a \tag{5}$$

(5) − (4) × 2 gives $5z = 2a$ or $z = \frac{2a}{5}$. From (4), $2y = a - z = a - \frac{2a}{5} = \frac{3a}{5}$ or $y = \frac{3}{10}a$. From (3), $3x + \frac{3a}{5} + \frac{4a}{5} = 2a$, i.e., $3x = 2a - \frac{7a}{5} = \frac{3a}{5}$ so that $x = \frac{a}{5}$.

SUPPLEMENT

The solutions in the earlier chapters boil down to solving polynomial equations of the form

$$x^2 + Ax + r = 0$$

$$x^3 + Ax^2 + Bx + r = 0$$

$$x^4 + Ax^3 + Bx^2 + Cx + r = 0$$

where $r > 0$ is the maximum value reached. We suggest algorithms for the solution in this supplement.

1. In the case of the quadratic, it is clear that the value of x is given by $x = -\frac{A}{2}$.

Example: $20x - x^2 = r$, i.e., $x^2 - 20x + r = 0$ giving $x = \frac{20}{2} = 10$.

2. Consider $x^3 + Ax^2 + Bx + r = 0$. Let a be the negative real root of the equation. Dividing by $x + a$, we have $x^3 + Ax^2 + Bx + r = (x + a)\{x^2 + (A - a)x + a^2 + B - aA\} + r - a(a^2 + B - aA)$.

So, $r = a(a^2 + B - aA)$ which gives the quadratic

$$x^2 + (A - a)x + \frac{r}{a} = 0.$$

For r to be maximum we should have

$$\frac{r}{a} = \frac{(A - a)^2}{4} = a^2 + B - aA \text{ so that}$$

$$A^2 - 2aA + a^2 = 4a^2 + 4B - 4aA$$

i.e., $3a^2 - 2aA = A^2 - 4B$ i.e., $a^2 - \frac{2}{3}aA = \frac{A^2 - 4B}{3}$.

This gives $x = -\frac{A-a}{2} = \frac{a-A}{2} = \frac{\sqrt{4A^2-12B}-2A}{6}$

$= 2\frac{\sqrt{A^2-3B}-2A}{6} = \frac{\sqrt{A^2-3B}-A}{3}$.

Example: In $x^3 - x^2 + r = 0$, $A = -1$, $B = 0$ so that

$$x = \frac{1+1}{3} = \frac{2}{3}.$$

In $x^3 - 6x^2 - 15x + r = 0$, $A = -6$, $B = -15$ so that

$$x = \frac{\sqrt{36+45}+6}{3} = \frac{15}{3} = 5$$

In $x^3 - x + r = 0$, $A = 0$, $B = -1$ so that

$$x = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$

. Consider $x^4 + Ax^3 + Bx^2 + Cx + r = 0$. If the product of wo of the linear factors of the biquadratic is $x^2 + ax + b$,

$x^4 + Ax^3 + Bx^2 + Cx + r$

$$\begin{aligned} = \quad & (x^2 + ax + b)\left\{x^2 + (A-a)x + B + a^2 - Aa - b\right\} \\ & + x\left\{(C + ab - Ab) - (aB + a^2 - Aa^2 - ab)\right\} \\ & + r - b(B + a^2 - Aa - b). \end{aligned}$$

Thus, we have

$$x^2 + (A-a)x + B + a^2 - Aa - b = 0 \qquad (1)$$

$$B + a^2 - Aa - b = \frac{r}{b} \qquad (2)$$

$$C + ab - Ab = aB + a^2 - Aa^2 - ab \qquad (3)$$

(3) gives

i.e., $b(2a-A) = aB + a^2 - Aa^2 - C$, i.e., $b = \frac{aB + a^2 - Aa^2 - C}{2a - A}$

If r is maximum, from (1) and (2), we have

$$\frac{r}{b} = \frac{(A-a)^2}{4} = B + a^2 - Aa - b,$$

i.e., $$\frac{(A-a)^2}{4} = B + a^2 - Aa - \frac{aB + a^2 - Aa^2 - C}{2a - A}$$

$$(A-a)^2 = \frac{4aB + 4a^3 - 8Aa^2 - 4AB + 4A^2a + 4C}{2a - A}$$

or $4aB + 4a^3 - 8Aa^2 - 4AB + 4A^2a + 4C = 4A^2a - 5a^2A + 2a^3 - A^3$

i.e., $$a^3 - \frac{3A}{2}a^2 + 2Ba - 2AB + 2C + \frac{A^3}{2} = 0.$$

If we can solve this cubic equation for a by inspection, then

$$x = -\frac{A-a}{2} = \frac{a-A}{2}.$$

Examples (i) $x^4 - x^3 + r = 0$. Here $A = -1,\ B = 0 = C$.

The cubic for a is $a^3 + \frac{3}{2}a^2 - \frac{1}{2} = 0.a = \frac{1}{2}$ is a solution.

$$x = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4}.$$

(ii) $x^4 - x + r = 0$. Here $A = 0,\ B = 0,\ C = -1$.

The cubic for a is $a^3 - 2 = 0$, i.e., $a = 2^{\frac{1}{3}}$,

$$x = \frac{2^{\frac{1}{3}}}{2} = \frac{1}{2^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{4}}.$$

(iii) $x^4 - 8x^3 + 22x^2 - 24x + r = 0$.

Here $A = -8,\ B = 22,\ C = -24$ so that the equation for a is $a^3 + 12a^2 + 44a + 48 = 0$. $a = -4$ is a root and

$$x = \frac{-4 + 8}{2} = 2.$$

(iv) To inscribe the parabola of greatest area in a given isosceles triangle. (Problem 7 of Chapter III). See Fig S.1.

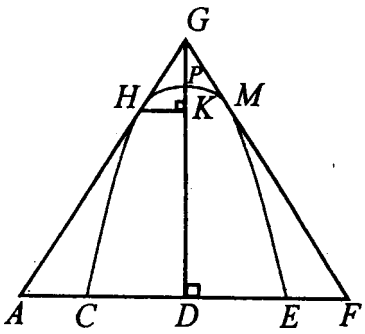


Fig. S.1

Solution: H, M are points of contact of the parabola with the sides GA, GF respectively of the isosceles triangle GAF, $GA = GF$. AD is the altitude from A. K is the foot of the perpendicular from H on GD. C, E are points of intersection of the parabola with AF. Area of the parabola $= \frac{4b}{3a}\sqrt{(a-x)^3 x}$ where $AD = b$, $GD = a$ (constants), $GP = x$ (variable). We are thus to maximize $(a-x)^3 x = a^3x - 3a^2x^2 + 3ax^3 - x^4$ or $x^4 - 3ax^3 + 3a^2x^2 - a^3x + r = 0$ is the required equation. Here $A = -3a$, $B = 3a^2$, $C = -a^3$. The equation to determine 'a' which we denote by y (to avoid confusion) is $y^3 + \frac{9a}{2}y^2 + 6a^2y + \frac{5a^3}{2} = 0$. $y = -\frac{5a}{2}$ is a root and

$$x = -\frac{A+y}{2} = \frac{3a - \frac{5a}{2}}{2} = \frac{a}{4}.$$

(v) Trapezium $ABCD$ has base $AB = a$, $AD = BC = b$. To find $CD \parallel AB$ such that the area $ABCD$ is maximum. (Fig S.2).

Solution: Let M, N be the feet of the perpendiculars from A, B respectively on CD. $AM = NB$, $DM = CN$. Le $CD = x$.

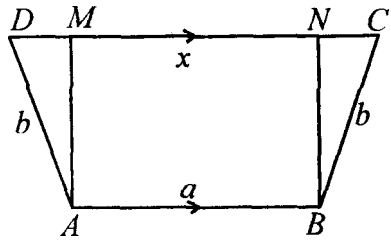


Fig. S.2

$$\begin{aligned}
\text{Area of } ABCD &= \frac{DM \times AM}{2} + MN \times AM + \frac{CN \times NB}{2} \\
&= \frac{(DM + CN)}{2} \times AM + MN \times AM \\
&\qquad\qquad (\text{since } AM = NB)) \\
&= (DM + MN) \times AM \text{ (since } DM = CN)) \\
&= \left(\frac{x-a}{2} + a\right) \times \sqrt{b^2 - \left(\frac{x-a}{2}\right)^2} \\
&= \frac{a+x}{2}\sqrt{b^2 - \left(\frac{a-x}{2}\right)^2} \\
&= \sqrt{\frac{b^2(a+x)^2}{4} - \left(\frac{(a^2-x^2)}{4}\right)^2}.
\end{aligned}$$

Thus, the equation to be considered is

$$x^4 - 2(2b^2 + a^2)x^2 - 8ab^2x - a^2(4b^2 - a^2) + r = 0.$$

Here $A = 0,\ B = -2(2b^2 + a^2),\ C = -8ab^2$.

The equation for 'a' denoted by y here is

$$y^3 - 4(2b^2 + a^2)y - 16ab^2 = 0.$$

$y = -2a$ is a root. Dividing by $y + 2a$ we have $y^2 - 2ay - 8b^2 = 0$ so that $y = a + \sqrt{8b^2 + a^2}$ and $x = \dfrac{y - A}{2} = \dfrac{y}{2} = \dfrac{a + \sqrt{8b^2 + a^2}}{2}$.

ADDITIONAL PROBLEMS

Problem 1. Find the greatest area which can be enclosed by four given straight line segments.

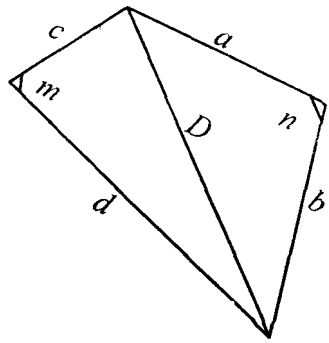


Fig. S.3

Solution: Let a, b, c, d be length of the given line segments. Let m, n be the angles included by the pairs (c, d); (a, b). (See Fig S.3).

$$\text{Area required} = \frac{cd \sin m}{2} + \frac{ab \sin n}{2} = \frac{cd}{2}\left(\sin m + \frac{ab}{cd} \sin n\right).$$

We are to maximize $\sin m + \frac{ab}{cd} \sin n$. Squaring

$$\sin^2 m + \frac{2ab}{cd} \sin m \sin n + \frac{a^2b^2}{c^2d^2} \sin^2 n = r, \quad \text{say.}$$

But $$c^2 + d^2 - 2cd \cos m = D^2 = a^2 + b^2 - 2ab \cos n \quad (1)$$

$$\cos^2 m - \frac{2ab}{cd} \cos m \cos n + \frac{a^2b^2}{c^2d^2} = b^2. \quad (2)$$

So, $\cos m - \frac{ab}{cd} \cos n = \frac{c^2 + d^2 - a^2 - b^2}{2cd} = B$.

Adding (1) and (2) and transposing

$$-\cos(m + n) = \frac{r + B^2 - \frac{a^2b^2}{c^2d^2} - 1}{2\frac{ab}{cd}}.$$

r is maximum when $\cos(m + n) = -1$, i.e., $m + n = 180°$. i.e., $ABCD$ is cyclic. Then Area $= \frac{ab + cd}{2} \sin n = \sqrt{(P - a)(P - b)(P - c)(P - d)}$. where $P = \frac{a + b + c + d}{2}$.

Problem 2. Find $x, y > 0$ such that $(mx + n)(ny + m)$ is maximum under the constraint $a^{mx}b^{ny} = c$, a, b, c being positive constants.

Solution: Taking logarithm, the constraint is $mx \log a + ny \log b = \log c$. Let $\log a = A$, $\log b = B$, $\log c = C$. Then the constraint is $mAx + nBy = C$. Hence $ny = \dfrac{C - mAx}{B}$ and so $ny + m = \dfrac{C - mAx + mB}{B}$.

Therefore $(mx + n)(ny + m)$

$$= \frac{-m^2Ax^2 + (mC + m^2B - mnA)x}{B} + \frac{nC + nmB}{B}.$$

So, $x^2 - \dfrac{(mC + m^2B - mnA)}{m^2A}x$ needs to be minimized.

The Solution is

$$\begin{aligned} x &= \frac{(mC + m^2B - mnA)}{2m^2A} \\ &= \frac{C + mB - nA}{2mA} \\ &= \frac{\log c + m \log b - n \log a}{2m \log a} \\ &= \frac{\log\left(\dfrac{cb^m}{a^n}\right)}{2m \log a}. \end{aligned}$$

Problem 3. OM and OP are two arcs of great circles on a given sphere and the arc PM is perpendicular to OM. Find when the difference between OP and OM is the greatest.

Solution: See Fig. S.4. Let $P\hat{O}M$ = angle between the planes in which arcs OP, OM lie = α. Let $OP = \phi$, $OM = \theta$ (i.e., angles subtended at the centre of the sphere). By Napier's formula for solution of right angled spherical triangles, $\tan\theta = \cos\alpha.\tan\phi$. Thus if

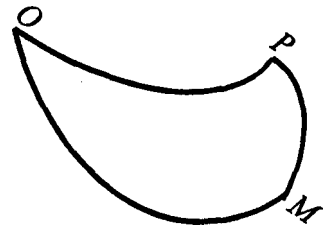


Fig. S.4

$$\theta = \phi - r,\ \tan\theta = \frac{\tan\phi - \tan r}{1 + \tan\phi\tan r} = \cos\alpha\tan\phi.$$

We are to maximize r or $\tan r$ governed by $\dfrac{x - r'}{1 + r'x} = \cos\alpha\tan\phi$ where $r' = \tan r$ and $x = \tan\phi$.

i.e., $$\frac{x - r'}{1 + r'x} = (\cos\alpha)x,$$

i.e., $$(\cos\alpha)x + r'(\cos\alpha)x^2 = x - r'.$$

i.e., $$(\cos\alpha - 1)x + r'(\cos\alpha)x^2 = r'.$$

i.e., $$-\frac{1 - \cos\alpha}{r'\cos\alpha}x + x^2 = \frac{1}{\cos\alpha}$$

i.e., $$\left(\frac{1 - \cos\alpha}{2r'\cos\alpha}\right)^2 + \left(x - \frac{1 - \cos\alpha}{2r'\cos\alpha}\right)^2 = \frac{1}{\cos\alpha}.$$

When r' is maximum $x = \dfrac{1 - \cos\alpha}{2r'\cos\alpha}$

and $$\left(\frac{1 - \cos\alpha}{2r'\cos\alpha}\right)^2 = \frac{1}{\cos\alpha}.$$

i.e., $$4r'^2\cos^2\alpha = \cos\alpha(1 - \cos\alpha)^2,$$

i.e., $$r'^2 = \frac{(1 - \cos\alpha)^2}{4\cos\alpha}.$$

Hence $$x = \frac{1 - \cos\alpha}{2 \times \dfrac{(1 - \cos\alpha)}{2\sqrt{\cos\alpha}}\cos\alpha} = \frac{1}{\sqrt{\cos\alpha}} = (\cos\alpha)^{\frac{-1}{2}}.$$

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